MATH 220 – SOLUTIONS TO PRACTICE FINAL

1. **(6)** Let $T(x_1, x_2) = (3x_1 + 2x_2, x_1, -x_1 + 4x_2)$ be a linear transformation.

   (a) Is T one-to-one? Justify your answer.

$$T(\bar{x}) = A\bar{x} \text{ where } A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ -1 & 4 \end{bmatrix} \xrightarrow[R_3+R_2]{R_1-3R_2} \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 4 \end{bmatrix} \xrightarrow[\text{swaps}]{R_3-2R_1} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

T is 1-to-1, as there is a pivot in every column.

   (b) Is T onto? Justify your answer.

T is not onto, as not every row has a pivot.

2. **(8)**

$$\text{Let } A = \begin{bmatrix} 1 & 3 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & 6 & 2 & 6 & 0 \end{bmatrix}. \text{ Then } \operatorname{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

pivot columns

   (a) Determine a basis for Col A.

A basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \\ 6 \end{bmatrix} \right\}$.

   (b) Determine a basis for Nul A.

Solutions to $A\bar{x} = \bar{0}$ are given by
$$\begin{aligned} x_1 &= -3x_2 + 4x_5 \\ x_3 &= 2x_5, \\ x_4 &= -2x_5 \end{aligned}$$
x_2, x_5 are free

i.e., $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 4 \\ 0 \\ 2 \\ -2 \\ 1 \end{bmatrix} x_5$, $x_2, x_5 \in \mathbb{R}$. A basis for Nul A is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

②

(c) What is dim Nul A? Explain.

dim Nul $A = 2$, as there are two free variables in $A\bar{x}=\bar{0}$.

(d) What is rank A? Explain.

rank $A = 3$, the # pivot columns in A.

3. **(10)** Let A and B be $n \times n$ matrices. We say that A and B are *similar* if there is an invertible matrix P such that $B = P^{-1}AP$. Show that if A and B^T are similar, then A and B have the same eigenvalues.

A and B^T similar $\Rightarrow$ there is an invertible $n\times n$ matrix P such that $A = P^{-1}B^TP$. → definition of similar matrices

$$\begin{aligned}
\Rightarrow \det(A-\lambda I) &= \det(P^{-1}B^TP - \lambda I) && \to \text{characteristic polynomial} \\
&= \det(P^{-1}B^TP - \lambda P^{-1}P) && \to P^{-1}P = I \\
&= \det\left(P^{-1}[B^TP - \lambda P]\right) && \to A(B+C) = AB+AC \\
&= \det(P^{-1}[B^T - \lambda I]P) && \to P = IP;\ (A+B)C = AC+BC \\
&= \det(P^{-1})\cdot\det(B^T-\lambda I)\cdot\det(P) && \to \det(AB) = \det A\cdot\det B \\
&= \frac{1}{\cancel{\det(P)}}\cdot\det\left[(B-\lambda I)^T\right]\cdot\cancel{\det(P)} && \to \det(A^{-1}) = \frac{1}{\det A};\ (A+B)^T = A^T+B^T \\
&= \det(B-\lambda I) && \to \det A^T = \det A;\ \det(P)\neq 0 \text{ if } P^{-1} \text{ exists}
\end{aligned}$$

Thus, A and B have the same characteristic polynomial, and hence the same eigenvalues.

4. **(10)** Let $A+B$ and C be $n\times n$ invertible matrices. Solve the following equation for X. Justify each step in your solution.

$$C\left(C^{-1}(XB + XA)C = C^T\right)$$

$$\Rightarrow \left(I(X(B+A))C = CC^T\right)C^{-1} \quad \to A(B+C) = AB+AC,\ CC^{-1} = I,\ C \text{ is invertible}$$

$$\left(X(A+B)\underbrace{CC^{-1}}_{I} = CC^TC^{-1}\right)(A+B)^{-1} \quad \to CC^{-1} = I,\ A+B \text{ is invertible}$$

$$X\underbrace{(A+B)(A+B)^{-1}}_{I} = CC^TC^{-1}(A+B)^{-1} \Rightarrow \boxed{X = CC^TC^{-1}(A+B)^{-1}}$$

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5. (8) The matrix $A = \begin{bmatrix} -1 & 3 & 3 \\ -3 & 5 & 3 \\ 3 & -3 & -1 \end{bmatrix}$ has eigenvalues $2, 2$ and -1. Determine a basis for the eigenspace corresponding to the eigenvalue $\lambda = 2$.

$(A - \lambda I)\bar{x} = \bar{0}$ for $\lambda = 2$

$$A - \lambda I = \begin{bmatrix} -3 & 3 & 3 \\ -3 & 3 & 3 \\ 3 & -3 & -3 \end{bmatrix} \xrightarrow[R_3 + R_1]{R_2 - R_1} \begin{bmatrix} -3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \times \frac{-1}{3}} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Bigg\} \begin{array}{l} x_1 = x_2 + x_3 \\ x_2, x_3 \text{ are free} \end{array}$$

Solutions are $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3$, $x_2, x_3 \in \mathbb{R}$. Hence, a basis for the eigenspace corresponding to the eigenvalue $\lambda = 2$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

6. (9) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 4 \\ -1 & 2 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

(a) If A is invertible, find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 + R_1]{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_2]{R_1 + R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 1 \end{array}\right] \xrightarrow[R_2 + 2R_3]{R_1 + 4R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -3 & 4 \\ 0 & 1 & 0 & 3 & -1 & 2 \\ 0 & 0 & -1 & 2 & -1 & 1 \end{array}\right] \xrightarrow{R_3 \times (-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -3 & 4 \\ 0 & 1 & 0 & 3 & -1 & 2 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array}\right]$$

Since A is row-equivalent to I_3, A is invertible, and $A^{-1} = \begin{bmatrix} 8 & -3 & 4 \\ 3 & -1 & 2 \\ -2 & 1 & -1 \end{bmatrix}$.

(b) If the inverse exists, use A^{-1} computed above to solve the system $Ax = b$.

$$\bar{x} = A^{-1}\bar{b} = \begin{bmatrix} 8 & -3 & 4 \\ 3 & -1 & 2 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix}.$$

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7. **(7)** Construct a *non-zero* 3×3 matrix A with rank 2, and a vector $\boldsymbol{b}$ that is *not* in Nul A.

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is a rank 2 matrix. We get a non-zero matrix from this matrix using ERO's

$\xrightarrow[R_3+R_1]{R_2+R_1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} = A.$ $\quad \bar{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is NOT in Nul A, as

$A\bar{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

8. **(8)** Let $\det A = 3$ and $\det B = 2$. Evaluate each of the following quantities, if possible. Justify your answers.

(a) $\det A^2 = (\det A)^2 = (3)^2 = 9.$ $\quad \det(A^k) = \underbrace{\det A \cdot \det A \cdots \det A}_{k} = (\det A)^k.$

(b) $\det(2AB^T) = (2)^n \cdot \det(A) \cdot \det(B)$ → cannot evaluate as n is not given, where A, B are $n \times n$.

$\det B^T = \det B.$

(c) $\det A^{-1} / \det B^{-1} = (1/\det A) \Big/ \left(\frac{1}{\det B}\right)$ $\quad \det A^{-1} = \frac{1}{\det A}$

$= \frac{1/3}{1/2} = \frac{2}{3}$

(d) $\det(A + B)$ → cannot evaluate! Specifically, $\det(A+B) \neq \det A + \det B$ typically.

9. **(7)** It is known that $x = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector of a 3×3 matrix A corresponding to the eigenvalue $\lambda = 0$. Is the linear transformation $T(x) = Ax$ one-to-one? Justify your answer.

$\bar{x}$ is an eigenvector corresponding to the eigenvalue $\lambda = 0$ implies $A\bar{x} = \lambda\bar{x} = \bar{0}$, and $\bar{x} \neq \bar{0}$. So, $\bar{x}$ is a non-trivial solution to $A\bar{x} = \bar{0}$. Hence there are free variables in $A\bar{x} = \bar{0}$, and hence, by IMT, $T(\bar{x}) = A\bar{x}$ is not 1-to-1.

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10. (8) Let $A = \begin{bmatrix} 2 & 5 \\ k & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$. What value(s) of k, if any, will make $AB = BA$?

$$AB = \begin{bmatrix} 2 & 5 \\ k & 1 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix} = \begin{bmatrix} 23 & 5k-10 \\ 4k+3 & -4k \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}\begin{bmatrix} 2 & 5 \\ k & 1 \end{bmatrix} = \begin{bmatrix} -5k+8 & 15 \\ k^2+6 & 15+k \end{bmatrix}$$

For $AB = BA$, we need

$15 + k = -4k \Rightarrow$

$k = \frac{-15}{5} = -3$

and

$5k - 10 = 15 \Rightarrow$

$k = \frac{25}{5} = 5$

not the same!

Hence $AB = BA$ for NO values of k.

11. (8) Let $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$.

(a) Is $\lambda = 1$ and eigenvalue of A? If yes, find an associated eigenvector.

$\det(A - \lambda I) = \begin{vmatrix} 1 & -4 \\ -1 & -2 \end{vmatrix} = -6 \neq 0$. Hence $(A - \lambda I)\bar{x} = \bar{0}$ has only the trivial solution. So $\lambda = 1$ is not an eigenvalue of A.

(b) Is $\lambda = -2$ and eigenvalue of A? If yes, find an associated eigenvector.

$A - \lambda I = \begin{bmatrix} 4 & -4 \\ -1 & 1 \end{bmatrix} \xrightarrow[R_2 \times (-1)]{R_1 + 4R_2} \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$ $\left.\right\}$ $x_1 = x_2$, x_2 free $\quad \bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$, $x_2 \in \mathbb{R}$

$\lambda = -2$ is an eigenvalue, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an associated eigenvector.

(c) Is $x = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ an eigenvector of A? If yes, find the corresponding eigenvalue.

$A\bar{x} = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ 3 \end{bmatrix} = 3\begin{bmatrix} -4 \\ 1 \end{bmatrix} = 3\bar{x}$. So $\bar{x}$ is an eigenvector corresponding to the eigenvalue $\lambda = 3$.

(d) Is $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ an eigenvector of A? If yes, find the corresponding eigenvalue.

$A\bar{x} = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \neq \lambda\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for any λ. Hence $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is not an eigenvector of A.

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12. (10) Decide whether each of the following statements is *True* or *False*. Justify your answer.

(a) If $Ax = b$ is inconsistent for some $b \in \mathbb{R}^n$, then $\lambda = 0$ is an eigenvalue of A.

TRUE. Notice that $\bar{b} \in \mathbb{R}^n$, so A is $n \times n$ here. $A\bar{x} = \bar{b}$ is inconsistent for some $\bar{b}$ means that A is not invertible (by IMT). Hence $A\bar{x} = \bar{0}$ has some non-trivial solution, say, $\bar{x}$. This $\bar{x}$ satisfies $A\bar{x} = \lambda\bar{x}$ for $\lambda = 0$.

(b) It could happen that $\det(A + B) = \det A + \det B$.

TRUE. Even though this statement is usually not true, consider $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$, $A+B = \begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}$.

Here $\det(A+B) = 0 = \det A + \det B = 0 + 0$.

(c) If x is an eigenvector of the matrix A corresponding to the eigenvalue λ, then $3x$ is an eigenvector corresponding to the eigenvalue 3λ.

FALSE. $A\bar{x} = \lambda\bar{x} \implies A(3\bar{x}) = 3(A\bar{x}) = 3(\lambda\bar{x}) = \lambda(3\bar{x})$.

So $3\bar{x}$ is also an eigenvector corresponding to the eigenvalue λ.

(d) If A is a 3×4 matrix, the largest value that $\dim \operatorname{Nul} A$ can take is 3.

FALSE. $\dim \operatorname{Nul} A$ can be 4, when A is the 3×4 zero matrix, i.e., when it has no pivot columns.

(e) If the system $Ax = b$ has more than one solution, then so does the system $Ax = \mathbf{0}$.

TRUE. $A\bar{x} = \bar{b}$ has free variables. We can perform the same EROs done on $[A|b]$ to solve this system on $[A|\bar{0}]$, and hence $A\bar{x} = \bar{0}$ also has free variables.