Introduction to Linear Algebra (Math 220_2) – Fall 2013
Practice Final

- There are **twelve** problems and **eight** pages in this exam.
- Show all work.
- Provide appropriate **justifications** where required.
- Good luck!

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1. **(6)** Let $T(x_1, x_2) = (3x_1 + 2x_2, x_1, -x_1 + 4x_2)$ be a linear transformation.
   (a) Is T one-to-one? Justify your answer.
   (b) Is T onto? Justify your answer.

2. **(8)**
$$\text{Let}\quad A = \begin{bmatrix} 1 & 3 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & 6 & 2 & 6 & 0 \end{bmatrix}. \quad \text{Then } \operatorname{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
   (a) Determine a basis for $\operatorname{Col} A$.
   (b) Determine a basis for $\operatorname{Nul} A$.
   (c) What is $\dim \operatorname{Nul} A$? Explain.
   (d) What is $\operatorname{rank} A$? Explain.

3. **(10)** Let A and B be $n \times n$ matrices. We say that A and B are *similar* if there is an invertible matrix P such that $B = P^{-1}AP$. Show that if A and B^T are similar, then A and B have the same eigenvalues.

4. **(10)** Let $A + B$ and C be $n \times n$ invertible matrices. Solve the following equation for X. Justify each step in your solution.
$$C^{-1}(XB + XA)C = C^T.$$

5. **(8)** The matrix $A = \begin{bmatrix} -1 & 3 & 3 \\ -3 & 5 & 3 \\ 3 & -3 & -1 \end{bmatrix}$ has eigenvalues $2, 2$ and -1. Determine a basis for the eigenspace corresponding to the eigenvalue $\lambda = 2$.

6. **(9)** Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 4 \\ -1 & 2 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.
   (a) If A is invertible, find A^{-1}.
   (b) If the inverse exists, use A^{-1} computed above to solve the system $A\mathbf{x} = \mathbf{b}$.

7. **(7)** Construct a *nonzero* 3×3 matrix A with rank 2, and a vector $\mathbf{b}$ that is *not* in $\mathrm{Nul}\, A$.

8. **(8)** Let $\det A = 3$ and $\det B = 2$. Evaluate each of the following quantities, if possible. Justify your answers.

   (a) $\det A^2$

   (b) $\det(2AB^T)$

   (c) $\det A^{-1} / \det B^{-1}$

   (d) $\det(A + B)$

9. **(7)** It is known that $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector of a 3×3 matrix A corresponding to the eigenvalue $\lambda = 0$. Is the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ one-to-one? Justify your answer.

10. **(8)** Let $A = \begin{bmatrix} 2 & 5 \\ k & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$. What value(s) of k, if any, will make $AB = BA$?

11. **(8)** Let $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$.

    (a) Is $\lambda = 1$ and eigenvalue of A? If yes, find an associated eigenvector.

    (b) Is $\lambda = -2$ and eigenvalue of A? If yes, find an associated eigenvector.

    (c) Is $\mathbf{x} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ an eigenvector of A? If yes, find the corresponding eigenvalue.

    (d) Is $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ an eigenvector of A? If yes, find the corresponding eigenvalue.

12. **(10)** Decide whether each of the following statements is *True* or *False*. Justify your answer.

    (a) If $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b} \in \mathbb{R}^n$, then $\lambda = 0$ is an eigenvalue of A.

    (b) It could happen that $\det(A + B) = \det A + \det B$.

    (c) If $\mathbf{x}$ is an eigenvector of the matrix A corresponding to the eigenvalue λ, then $3\mathbf{x}$ is an eigenvector corresponding to the eigenvalue 3λ.

    (d) If A is a 3×4 matrix, the largest value that $\dim \mathrm{Nul}\, A$ can take is 3.

    (e) If the system $A\mathbf{x} = \mathbf{b}$ has more than one solution, then so does the system $A\mathbf{x} = \mathbf{0}$.