MATH 220-2 SOLUTIONS TO MIDTERM EXAM

1. (a) $$\begin{bmatrix} 0 & 5 & -1 \\ 2 & 0 & 9 \\ 1 & -4 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}.$$

(b) $$x_1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 0 \\ -4 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 9 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}.$$

(c) $$\left[\begin{array}{ccc|c} 0 & 5 & -1 & 7 \\ 2 & 0 & 9 & 0 \\ 1 & -4 & 0.5 & 3 \end{array}\right].$$

2. (a) $$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{array}\right] \xrightarrow{R_3 \rightleftharpoons R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \end{array}\right] \xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - R_2} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 = 0$
$x_2 + x_3 = 2$
x_3 free

Solutions are $$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} s, \ s \in \mathbb{R}.$$

(b) Solutions to $A\bar{x} = \bar{0}$ are $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} s$, $s \in \mathbb{R}$.

Solutions to $A\bar{x} = \bar{b}$ are obtained by adding a constant vector to the solutions of $A\bar{x} = \bar{0}$. In this case, this vector is $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.

3. The system is $A\bar{x} = \bar{b}$, where $A = [\bar{u} \ \bar{w}]$, and $\bar{b} = \bar{v}$. Hence we want x_1, x_2, not both zero, such that $x_1 \bar{u} + x_2 \bar{w} = \bar{v}$.

$$4\bar{u} - 3\bar{v} - \bar{w} = 0 \implies 4\bar{u} - \bar{w} = 3\bar{v} \implies \left(\frac{4}{3}\right)\bar{u} + \left(\frac{-1}{3}\right)\bar{w} = \bar{v}.$$

Hence, $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$ is a nontrivial solution.

4. We can construct a non-zero 3×3 matrix whose columns span $\mathbb{R}^3$. Then the given $\bar{b}$ will also be in its span.

We could start with $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, whose columns span $\mathbb{R}^3$, as there is a pivot in every row. We can then use EROs to convert the 0's to non-zeros.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2+R_1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}.$$

Thus, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$ will have $\bar{b}$ in the span of its columns.

5. (a) $$\begin{bmatrix} 4 & 2 & -2 & 3 \\ 1 & 6 & 5 & 0 \\ 3 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \rightleftarrows R_2} \begin{bmatrix} 1 & 6 & 5 & 0 \\ 4 & 2 & -2 & 3 \\ 3 & 1 & 1 & 2 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2-4R_1} \begin{bmatrix} 1 & 6 & 5 & 0 \\ 0 & -22 & -22 & 3 \\ 0 & -17 & -14 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3-\left(\frac{17}{22}\right)R_2} \begin{bmatrix} 1 & 6 & 5 & 0 \\ 0 & -22 & -22 & 3 \\ 0 & 0 & 3 & -7/22 \end{bmatrix}$$

Pivot in every row. So $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ spans $\mathbb{R}^3$.

(b) No. We need at least 3 LI vectors to span $\mathbb{R}^3$. The set $\{\bar{v}_1, \bar{v}_2\}$ has only two. So, $A = [\bar{v}_1\ \bar{v}_2]$ cannot have a pivot in every row.

6\. (a) $A=\begin{bmatrix} 2 & 3 \\ 1 & k \\ h & 0 \end{bmatrix}$ $T(\bar{x})=A\bar{x}$ is 1-to-1 if A has a pivot in every column.

$$\begin{bmatrix} 2 & 3 \\ 1 & k \\ h & 0 \end{bmatrix} \xrightarrow{R_2 \rightleftarrows R_1} \begin{bmatrix} 1 & k \\ 2 & 3 \\ h & 0 \end{bmatrix} \xrightarrow[R_3-hR_1]{R_2-2R_1} \begin{bmatrix} 1 & k \\ 0 & 3-2k \\ 0 & -hk \end{bmatrix}$$

The only way the second column will **NOT** be a pivot column is when both $3-2k=0$ and $-hk=0$, i.e., when $k=\frac{3}{2}$, $h=0$.

So, T is 1-to-1 for all real values h, k, except for the pair $h=0$, $k=\frac{3}{2}$.

(b) T cannot map $\mathbb{R}^2$ onto $\mathbb{R}^3$, as we need a pivot in every row of A, but A can have at most two pivots.

7\. T is not linear, as $T(\bar{0}) \neq \bar{0}$.

$T(0,0) = (2\times 0,\ 3\times 0-5\times 0,\ -0+1) = (0,0,1) \neq 0$.

8\. (a) FALSE. Notice that $\bar{b}\in\mathbb{R}^n$ here, hence A is $n\times n$ (square). If $A\bar{x}=\bar{b}$ is inconsistent, then there cannot be a pivot in every row. Hence, there must be at least one free variable (as the # rows = # columns = n here). So, $A\bar{x}=\bar{0}$ has non-trivial solutions.

(b) FALSE. $\bar{b}$ may not even be of the same dimension as $\bar{x}$ to start with! For instance, in the system $x_1-4x_2+5x_3=4$, $\bar{b}$ is a 1-vector, while all solutions are 3-vectors $\bar{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

(c) FALSE. With $\bar{v}_1=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\bar{v}_2=\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, $\{\bar{v}_1,\bar{v}_2\}$ is LD.

(d) FALSE. We can have pivots in every column without having a pivot in every row, e.g., take $A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.