A Model for Granular Statics with Impenetrability Constraints

K. A. Ariyawansa, Leonid Berlyand and Alexander Panchenko

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Abstract. We study the effects of geometric impenetrability constraints in statics of frictionless granular packings. The packing is deformed by imposing boundary conditions, modeling those in shear and compression experiments. In our two-dimensional model, the packing is represented by a spring-lattice network, whereby the particle centers correspond to vertices of the network, and interparticle contacts correspond to the edges. For the springs, we use a quadratic elastic interaction potential. Combined with the linearized impenetrability constraints, it provides a regularization of the hard-sphere potential for small displacements. When the network deforms, each spring either preserves its length (this corresponds to a solid-like contact), or expands (this represents a broken contact). Solid-like contacts are either sheared or stuck. A contact between two particles is sheared when the local motion of one particle relative to another is an infinitesimal shear. If a pair of particles in contact moves as a single rigid body, the contact is stuck.

Our goal is to study distribution of solid-like contacts in the minimizing configuration. We prove that under certain geometric conditions on the network, there are at least two non-stretched springs attached to each node, which means that every particle has at least two solid-like contacts. We also give a geometric criterion restricting the choice of these contacts.

Key words: granular materials, constrained optimization, geometric rigidity, discrete variational inequalities.

1. Introduction

Materials that are composed of collections of separate, macroscopic solid grains belong to the general classification of granular materials. Examples of such materials are common, including sand, gravel, medicinal pills, coins, and breakfast cereal. Granular media are important to numerous industries ranging from mining to pharmaceuticals. In geophysics, granular materials are a central problem in understanding the physics of earthquakes and tectonic faulting. Earthquake fault zones produce granular wear material continuously as a function of shear and grinding between the fault surfaces. The wear material, known as fault gouge, varies in thickness from 10's of cm to 1000 m and plays a critical role in determining the fault zone frictional strength, the stability of fault slip, and the size of the rupture nucleation dimension.

Granular media display a variety of complex static and dynamic properties that distinguish them from conventional solids and liquids. The complexity of granular media lies primarily in the collective properties of a macroscopic number of grains...
and how they interact with each other. The conditions under which a granular medium is stable or flows and the nature of this flow depend critically on the distributions of grain size and shape as well as the interactions between the grains. The practical importance of granular media combined with the richness of their physical properties has led to a great deal of interest from both theoretical and experimental points of view [4, 10, 11].

An important class of granular materials consists of nearly rigid particles that possess the following property: if a moderate force is applied, the particles start to move, and only after a substantial increase of the force the particles deform significantly. In other words, for loads which are not very high, the deformations inside a particle are small compared with the displacement of the particle center of mass. Consequently, the particle shapes change very little, so that each particle can be associated with a region of space that is inaccessible to any other particle. This gives rise to constraints on the admissible positions of particles. These impenetrability constraints are also known as geometric, kinematic, and excluded volume constraints.

A physical phenomenon related to appearance of constraints is jamming. A particle is jammed when its motion is completely obstructed by the neighbors, so the whole cluster of neighboring particles can only move together as a rigid body. The corresponding mathematical notion of rigidity ([22]) can be applied to various physical objects, such as sphere packings, and frameworks (trusses), as well as mathematical objects. In particular, an important mathematical object associated with any particle packing is the contact graph. Vertices of this graph are particle centers, while edges represent interparticle contacts.

The simplest physical model that exhibits jamming is a classical hard sphere packing. The particles in this model are represented by rigid spheres, and the only interparticle forces are reactions of constraints. Rigidity of hard sphere packings is studied in [2]. This problem can be formulated as a problem of detecting rigidity of the cable framework associated with the contact graph of the packing. The framework is obtained by replacing edges of the contact graph with the cables, and vertices with flexible hinges. The lengths of the cables are not allowed to decrease, which models the impenetrability constraints. Recently, a linear programming algorithm for detecting rigidity in hard sphere packings (equivalently, cable frameworks) was proposed in [5]. A cable framework is a special case of the so-called tensegrity frameworks studied in [3].

Another special case important in the present work is a bar framework. A bar framework is obtained from a graph by replacing the edges with rigid bars, and vertices with hinges. A bar framework and the associated graph are called rigid if the only possible vertex motions correspond to rigid motions of the whole framework. We note that both bar and cable frameworks can be associated to the same graph. To generate the bar framework, the edges are replaced with rigid bars that can only translate and rotate. In the case of the cable framework, one replaces edges with cables that can either move as rigid bodies or stretch. Thus every motion of a bar framework is also permitted by the cable framework, but the converse is not true in general. Therefore, it is possible that a bar framework associated to a graph is rigid, while the cable framework corresponding to the same graph is flexible.

It appears that the only currently available mathematical results [2, 3, 5] on discrete particle systems with geometric constraints concern hard particle packings.
To the best of authors’ knowledge, there are no results on frictional packings, and even elastic frictionless packings have not been yet studied. The present work differs from [2, 3, 5] in several respects. First, all these studies deal with rigid particles. We consider a more realistic situation of particles with elastic interactions, defined by a quadratic potential energy. Second, we address a different question. We are interested in generic contact patterns of the the energy minimizing configurations, while [2, 3, 5] focus on jamming. The packings that we study are not jammed. Their contact graphs are such that the associated bar framework is rigid, while the associated cable framework is flexible. The third difference is in the type of the boundary conditions. The conditions in [5] are periodic or hard wall conditions. The periodic conditions are commonly used to minimize influence of the boundaries in the problem. However, presence of walls is a major factor that determines bulk behavior of granular materials, and therefore it seems better to use boundary conditions that occur in engineering and physical experiments, where the walls are rigid and typically moving. Since our approach uses realistic boundary conditions, it is well suited for studying finite size systems that occur in experiments.

One of frequently observed properties of granular materials is concentration of the bulk deformation in thin layers called shear bands. For quasi-static flows driven by small shear rates, the corresponding patterns are called micro-bands ([14]) or network of weak contacts ([19]). The goal of this paper is to describe some generic geometric features of such patterns in dense packings of nearly rigid particles. Pattern formation may be caused by the local jamming (which mathematically amounts to impenetrability constraints), and friction. Since dealing with friction is difficult, and no results on frictionless elastic packings are currently available, we consider a model in which friction is neglected, but impenetrability constraints are retained, at least as a means of determining the relative importance of friction and constraints in formation of micro-bands.

In this paper, we consider a dense packing of elastic deformable particles without friction. In two dimensions, particles are represented by disks \( D_i \) of radii \( a_i \) with centers \( x_i, i = 1, 2, \ldots, N \). The initial reference configuration is deformed by applying prescribed small displacements to the boundary particles. Assuming that the deformations inside of the individual particles are small, and neglecting rotational degrees of freedom, one can characterize the deformations of \( D_i \) by the displacements \( \mathbf{u}^i \) of their centers. The elastic interaction forces are modeled as in classical mechanics of point particles: the force exerted by \( D_j \) on \( D_i \) is applied at \( x^i \), its direction is along the line joining \( x^i \) and \( x^j \), and its magnitude depends linearly on \( \mathbf{u}^j \)

We further assume that the granular material is pre-stressed (or, equivalently, the material is under confining stress). This means that in the reference state, the particles are squashed into each other as a result of applied external pressure. Further compression is supposed to be impossible (requires infinite energy), which introduces impenetrability constraints into the problem. To model impenetrability, one can, for instance, require that

\[
| (x^i + \mathbf{u}^i) - (x^j + \mathbf{u}^j) | \geq a_i + a_j,
\]

for each pair of particles. Since the the packing is dense, and the displacements are expected to be small, it makes sense to require that a particle cannot escape a cage formed by its neighbors. Therefore, the contacts that exist in the reference configuration may be broken, but no new contacts are created during the deformation. An
important consequence of this assumption is as follows. If the deformation satisfies
the constraints for each pair of particles in contact, then it automatically satisfies
all constraints. In the sequel, we use this assumption in the course of proving the
main result of this paper (Theorem 5.1).

Next, we introduce a network model which describes a granular material under
the above assumptions. The vertices of the network are the particle centers, and the
edges represent particle contacts. The collection of vertices \( x^i, i = 1, 2, \ldots, N \), and
edges forms the contact network (graph) \( \Gamma \). We suppose that \( \Gamma \) is a triangulation
of a connected, convex polygonal domain \( \Omega \). Another natural triangulation generated
by \( x^i, i = 1, 2, \ldots, N \) is the Delaunay graph \( G \). In principle, \( \Gamma \) and \( G \) may be
different. In general, \( \Gamma \) is a subgraph of \( G \), and the difference between the two
graphs is larger for relatively loose packings. In the present case, we suppose that
\( \Gamma \) and \( G \) coincide, which corresponds to “maximally dense” packings.

For small displacements \( u^i \), the quadratic constraints (1.1) can be approximated
by their linearizations near \( u^i = 0 \), which leads to the linearized impenetrability
constraints
\[
(u^j - u^i) \cdot q^{ij} \geq 0, \quad i = 1, 2, \ldots, N
\]
for each pair of vertices \( i, j \) connected by an edge of \( \Gamma \). In (1.2), \( q^{ij} = (x^j - x^i)/|x^j - x^i| \) are unit vectors that point from \( x^i \) to \( x^j \) along the line of centers.
Note that if the position of \( D_i \) is fixed \( (u^i = 0) \), then \( u^j \) satisfying (1.2) must lie
in the half-plane \( u \cdot q^{ij} \geq 0 \), so that \( D_j \) would be moving away from \( D_i \).

All the contacts must satisfy (1.2), but we further distinguish two types of contacts: broken and solid-like (see Fig. 1). We call a contact broken if
\[
(u^j - u^i) \cdot q^{ij} > 0,
\]
and solid-like if
\[
(u^j - u^i) \cdot q^{ij} = 0.
\]

The solid-like contacts correspond to two possible types of pair motions. The
first type is a rigid motion of a pair, in which case the contact is called a stuck
contact.

![Figure 1](image.png)

**Figure 1.** a) A broken contact. On the left—reference configuration, on the right—displaced configuration in the local coordinates of the bottom particle; b) A solid-like contact. The figure corresponds to an infinitesimal rolling of the top particle.

The second type is a local shear motion. In the local coordinates of one particle,
it is either the motion of the second particle in the direction perpendicular to the
line of centers, or an infinitesimal rotation (rolling). The corresponding contacts are
called sheared. In our idealized model, friction is neglected, and any tangential force
would lead to immediate separation of particles, because for disk-shaped particles, the contact surface is a point. We, however, still call these contacts solid-like, because in reality, these contacts are subject to friction forces, the contact surface has a positive area, and the particles in a sheared contact will not separate until the tangential force reaches the static friction threshold. In the case of rolling, the particles stay in contact and the pair is capable of bearing a compressive load.

Physically, vertices of the network can be realized as unit point masses and edges as elastic springs. Elastic force of the spring $(i,j)$ is determined by the pair potential $H(t_{ij})$, where $t_{ij} = (\mathbf{u}^j - \mathbf{u}^i) \cdot \mathbf{q}^{ij}$. The potential is a key ingredient of our model, and therefore we discuss it in detail. To motivate the choice of $H$, we first recall the classical hard sphere potential $H_{hs}$, which in our notation is defined by

$$
H_{hs}(t_{ij}) = \begin{cases} 
\infty & \text{if } t_{ij} < 0, \\
0 & \text{if } t_{ij} \geq 0.
\end{cases}
$$

$H_{hs}$ models the following two options: (i) moving non-deformable (hard spheres) particles toward each other requires infinite energy (a vertical line at $t_{ij} = 0$), (ii) moving particles apart requires no energy. Note that (1.5) already incorporates the constraints (1.2) by imposing an “infinite penalty” if the constraints are violated.

$$
H(t_{ij}; d) = \begin{cases} 
\infty & \text{if } t_{ij} < 0, \\
\frac{1}{2}d^{-3}(t_{ij} - d)^2 & \text{if } t_{ij} \geq 0.
\end{cases}
$$

The potential (1.6) is shown on Fig. 2, together with the hard-sphere potential. The formula (1.6) describes two options: (i) moving particles toward each other requires infinite energy; (ii) movement of particles apart from each other is caused by finite, linear elastic force, corresponding to a quadratic potential; This force is repulsive for small distances ($t_{ij} < d$). The magnitude of the force $f(t_{ij}, d) = \left| \frac{\partial H(t_{ij}, d)}{\partial t_{ij}} \right| = \frac{1}{2}d^{-3}|t_{ij} - d|$. The magnitude of the ambient force of pre-stress (or simply the confining pressure) is given by $\lim_{t_{ij} \to 0^+} f(t_{ij}, d) = \frac{1}{2}d^{-2}$, which tends to zero as $d \to \infty$. Therefore, the effect of pre-stress is smaller for larger $d$. Further, $H(t_{ij}, d)$ regularizes $H_{hs}$ in the following sense: if $d \geq d_0 > 0$, then $\lim_{d \to \infty} H(t_{ij}, d) = 0$ uniformly on $(0, d_0]$. In the paper, we do not pass to this limit. Instead we choose $d$ sufficiently large and fix it. In reality, once the distance between $D_i$ and $D_j$ is
greater than the sum of their radii \( a_i + a_j \), the pair interaction force is zero. In our model, we still have a small repulsive force for all \( a_i + a_j \leq t_{ij} \leq d \). So, the particles in our model would continue accelerating away from each other even after separating. This encourages separation of particles, and could lead to increase in the number of broken contacts. Since our goal is estimating the number of solid-like contacts from below, this “extra repulsive force” seems acceptable. We also mention that elastic contact force predicted by the classical Hertz theory is a nonlinear function of \( t_{ij} \). Our model is chosen for simplicity, and can be viewed as an approximation of Hertz theory, valid for sufficiently small displacements.

Next, there is no reason to choose the regularization parameter \( d \) the same for all pairs of particles in contact. Therefore, we define the pair interaction energy

\[
h(t_{ij}, d_{ij}) = \frac{1}{2} d^{-3} (t_{ij} - d_{ij})^2,
\]

and introduce the following generalization of the formula (1.6):

\[
H(t_{ij}, d_{ij}) = \begin{cases} 
\infty & \text{if } t_{ij} < 0, \\
h(t_{ij}, d_{ij}) & \text{if } t_{ij} \geq 0,
\end{cases}
\]

where

\[
d_{ij} = d_{i\delta ij}, \delta_{ij} \in [1/2, 1].
\]

This choice of \( d_{ij} \) ensures that the vertices of all pair quadratic potentials are positive and located in the interval \([1/2d, d]\). Finally, the total elastic interaction energy \( Q \) of the network is obtained by summing up \( h(t_{ij}, d_{ij}) \) over all pairs \((i, j)\) corresponding to the edges of the network.

The main problem addressed in the paper is minimization problem for \( Q \) subject to the constraints (1.2) and the appropriate boundary conditions. Since the functional \( Q \) is quadratic, and the constraints and boundary conditions are linear, this problem is a classical quadratic programming, extensively studied in optimization theory [9]. In the language of optimization theory, the solid-like contacts (1.4) correspond to the so-called active constraints, while the constraints corresponding to the broken contacts (1.3) are called inactive. The question studied in this paper concerns the number and spatial distribution of each type of constraints in the energy-minimizing configuration of the network. It appears that no general results of this type are currently available in optimization theory. The present study makes use of the geometric features of the contact graph, in particular its rigidity properties, to investigate the energy minimizer. We also note the connection between our constrained variational problem and continuum variational inequalities [6], [12], [13], [16]. Our problem can be viewed as a discrete variational inequality.

The main result of the paper is Theorem 5.1 where we prove that the constrained energy minimizer corresponds to a packing with at least two solid-like contacts per each particle. The network of solid-like contacts is the load-bearing structure, while broken contacts can be associated with the so-called micro-bands [14], [15] that appear during small shear deformations. The result implies that no particle can lose contact with all of its neighbors, which eliminates “micro-avalanches”. Put another way, loss of structural integrity in dense packings is evolutionary rather than catastrophic, so that shearing with a small displacement will first lead to dilatation, during which the packing becomes more loose everywhere, and only then local avalanches may occur.
Another useful consequence of theorem 5.1 is as follows. It provides a lower bound on the order parameter, recently introduced in [1, 21] as one of the main ingredients of the new phenomenological theory of dense granular flows. In [1, 21], the order parameter characterizes the phase transition of a granular material from solid-like to fluidized state. The informal definition of the order parameter in [21] is as follows. The order parameter at a point is the ratio of the number of solid-like contacts to the number of all contacts. A contact is considered solid-like [21] if two particles are jammed together for longer than a characteristic collision time. Both numbers are averaged over a certain control volume containing the point. Based on our analysis of different types of contacts, we suggest a slightly different definition of the order parameter: it is a ratio of the number of all solid-like contacts (both stuck and sheared) to the number of all contacts. This definition is natural from the mathematical point of view, since in optimization theory solid-like contacts correspond to the so-called active constraints. It is also natural from the point of view of physics, since sheared contacts in the actual granular packing will be subject to friction, and thus partially stable. Recent results of numerical simulations in [7, 8] suggest that friction enhances bulk elastic properties of granular media.

The paper is organized as follows. In Sect. 2 we formulate the main constrained minimization problem. Elimination of the equality constraints corresponding to the boundary conditions leads to a reduced minimization problem. In Sect. 3 we recall some facts concerning first-order rigidity of graphs. In Sect. 4 we show existence of a unique minimizer of the reduced problem. Optimality conditions for the reduced problem are stated and analyzed in Sect. 5. Here we state and prove the main theorem 5.1. In Sect. 6 we introduce a definition of the order parameter in the spirit of [21] and give a lower bound on the order parameter that follows from the main theorem. Finally, conclusions are provided in Sect. 7.

2. Formulation of the problem

2.1. Elastic interactions with impenetrability constraints. In 2D, consider a packing of spheres $D_i$ of radii $a_i$, with centers $x^i$, $i = 1, 2, \ldots, N$. (All vectors in this paper are column vectors and we use superscript $'$ to indicate transposition.) The packing fills a bounded region. After an infinitesimal motion, the position of the center of $D_i$ is $y^i$. We write $y^i = x^i + u^i$ where $u^i$ are displacements. The vertices $x^i, x^j$ are connected by an edge if and only if $D_i, D_j$ are in contact. In this case, we call $x^i$ and $x^j$ neighbors. We denote by $N_i$ the set of $j \in \{1, 2, \ldots, N\}$ such that $x^j$ is a neighbor of $x^i$. Orientation of contacts (equivalently, edges) is prescribed by the unit vectors

\[
q^{ij} = \frac{x^j - x^i}{|x^j - x^i|}.
\]

The vertices $x^i$ and edges $(i, j)$ define the contact graph $\Gamma$. Let $E$ denote the number of edges of $\Gamma$. The edge set $E$ of $\Gamma$ is given by $\{(i, j) : j \in N_i, i = 1, 2, \ldots, N\}$. To each edge $(i, j)$ we can associate a pair potential energy $h(t_{ij}, d_{ij})$ defined in (1.7). Summing up all these energies we obtain the total elastic interaction energy of the network. It is a quadratic form

\[
Q(u^1, u^2, \ldots, u^N) = \sum_{i=1}^{N} \sum_{j \in N_i} h(t_{ij}, d_{ij}) = \frac{1}{2} d^{-3} \sum_{i=1}^{N} \sum_{j \in N_i} ((u^j - u^i) \cdot q^{ij} + d_{ij})^2,
\]
on the displacements $u^i, i = 1, 2, \ldots, N$. In (2.2), $d, d_{ij}$ are parameters specified by (1.8), (1.9).

Our objective is to determine the displacements $u^i, i = 1, 2, \ldots, N$ so that the energy functional $Q$ is minimized subject to two types of constraints. The first type of constraints consists of linearized impenetrability constraints. These are obtained by formally linearizing the condition that the distance between two spheres in contact cannot decrease. Consider two spheres $D_i, D_j$ in contact. In the reference configuration,

$$\|x^i - x^j\| = a_i + a_j.$$  

(2.3)

Assuming that $D_i, D_j$ cannot overlap, we have

$$\|y^i - y^j\| \geq a_i + a_j.$$  

(2.4)

These are the impenetrability constraints. We linearize (2.4) by writing

$$\|y^i - y^j\|^2 = \|x^i - x^j - u^i + u^j\|^2 = \|x^i - x^j\|^2 + 2(x^i - x^j) \cdot (u^i - u^j) + \|u^i - u^j\|^2 = (a_i + a_j)^2 + 2(x^i - x^j) \cdot (u^i - u^j) + \|u^i - u^j\|^2.$$  

Now for “small” $\|u^i - u^j\|$ we can neglect quadratic term $\|u^i - u^j\|^2$, and (2.4) yields $2(x^i - x^j) \cdot (u^i - u^j) \geq 0$, which in turn is equivalent to $(u^j - u^i) \cdot q^{ij} \geq 0$ where $q^{ij}$ is as defined in (2.1). Therefore, the first set of constraints we impose on the displacements $u^1, u^2, \ldots, u^N$ is

$$\begin{align*} 
(u^j - u^i) \cdot q^{ij} &\geq 0, \quad j \in \mathcal{N}_i, \quad i = 1, 2, \ldots, N. 
\end{align*}$$  

(2.5)

The second type of constraints corresponds to the boundary conditions. Particles located at the packing boundary have prescribed displacements. In the sequel we refer to these particles as boundary particles. The corresponding vertices of $\Gamma$ are called boundary vertices. Other particles are referred to as interior, or sometimes, free, and the corresponding vertices of $\Gamma$ as interior vertices.

All boundary particles are divided into several groups, numbered $1, 2, \ldots, M$. Let $I_m$ denote the set of indices of the particles in group $m$ for $m = 1, 2, \ldots, M$. Each sphere in a certain group is in contact with at least one other sphere from the same group. Each group moves as a single rigid body. We assume that the prescribed boundary displacements are of the form

$$u^i = R^m(x^i), \quad i \in I_m, \quad m = 1, 2, \ldots, M,$$  

(2.6)

where

$$R^m(x^i) = c^m + \alpha^m K(x^i - x^{*m}), \quad i \in I_m, \quad m = 1, 2, \ldots, M,$$  

(2.7)

and $c^m, x^{*m}$ are given vectors, $\alpha^m$ is a given scalar, and $K$ is the matrix denoting clockwise rotation by $\pi/2$. The functions $R^m$ are called infinitesimal rigid deformations, parametrized by a scalar $\alpha^m$, and vectors $c^m$ and $x^{*m}$. We refer the reader to Sect. 3 for more details on rigid displacements.
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Our description above leads to the

**Main problem:**

\begin{align}
(2.8) \quad & \text{minimize} \quad Q(\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^N) \\
(2.9) \quad & \text{subject to} \quad \text{linearized impenetrability constraints (2.5)} \\
(2.10) \quad & \text{and boundary conditions (2.6)}.
\end{align}

2.2. **Feasible region.** Let us define the configuration space $U$. Points of this space are denoted by $U = ((\mathbf{u}^1)^T, (\mathbf{u}^2)^T, \ldots, (\mathbf{u}^N)^T)^T$.

**Remark.** To avoid this heavy notation, we simply write $U = (\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^N)$ when no confusion can occur.

Dimension of $U$ is $2N$. **Feasible region** $\mathcal{F}$ is the subset of $U$ in which all the constraints (2.5) and (2.6) are satisfied. The points satisfying (2.5) form a polyhedral (not necessarily bounded) region. The boundary of this region consists of parts of the hyperplanes (subspaces of dimension $2N - 1$) defined by

\begin{equation}
(\mathbf{u}^j - \mathbf{u}^i) \cdot \mathbf{q}^{ij} = 0, \quad j \in \mathcal{N}_i, \quad i = 1, 2, \ldots, N.
\end{equation}

Because of the connections with rigidity, we refer to (2.11) as $R$-equations. Equations (2.6) define $M$ planes $S_m, m = 1, \ldots, M$. Dimensions of $S_m$ depend on the number of constrained spheres in the $m$-th group.

For each point of $U \in \mathcal{F}$, some of the constraints (2.5) are satisfied as equations. These constraints are called active. The corresponding edges of the contact graph $\Gamma$ are called active as well. The rest of (2.5) are satisfied as strict inequalities. These are inactive constraints (respectively, edges).

2.3. **Elimination of constraints corresponding to boundary conditions.** The quadratic form $Q$ in (2.2) can be written in a convenient form in terms of a certain matrix $R^r$. To define $R^r$, we index the edges of $\Gamma$ by $l, l = 1, 2, \ldots, E$. Let $(i_l, j_l) \in \mathcal{E}$ be the edge of $\Gamma$ corresponding to $l$ for $l = 1, 2, \ldots, E$. Let $R^r$ be the $E \times 2N$ matrix whose $l$-th row is defined by

\begin{equation}
R^r_{lm} = \begin{cases} 
+q^{i_l j_l} & \text{if } m = 2(j_l - 1) + 1 \\
+q^{i_l j_l} & \text{if } m = 2(j_l - 1) + 2 \\
-q^{i_l j_l} & \text{if } m = 2(i_l - 1) + 1 \\
-q^{i_l j_l} & \text{if } m = 2(i_l - 1) + 2 \\
0 & \text{otherwise}
\end{cases}
\end{equation}

for $l = 1, 2, \ldots, E$.

**Remarks.** 1. $R^r$ is the (first-order) rigidity matrix, a well known object in geometric rigidity theory (see e.g. [3, 22]).

2. Consider vertices $\mathbf{x}^{i_l}, \mathbf{x}^{j_l}$ and the edge $l$ connecting them. The corresponding row $r^l$ of $R^r$ has $2N$ entries. We can view $r^l$ as a string of $N$ pairs of numbers, the first pair corresponding to $\mathbf{x}^1$, the second to $\mathbf{x}^2$ and so on. For simplicity, we shall call a pair of entries corresponding to a particular vertex $\mathbf{x}^i$ a place corresponding to $\mathbf{x}^i$.

Then we can interpret equation (2.12) as follows. A row $r^l$ has zeros at all places, except two. The non-zero entries are $-q^{i_l j_l}$, written as a two-dimensional
row vector at the place corresponding to $x^i$; and $q^{i:j}$, written as a two-dimensional row at the place corresponding to $x^j$.

3. A row of $R^r$ corresponds to an edge of $\Gamma$. Therefore it is natural to call a row active (respectively, inactive) if a corresponding edge is active (respectively, inactive).

Now define the vector $d_\text{E}$ by

$$d_\text{E} = \left( d_{i_1;j_1}, d_{i_2;j_2}, \ldots, d_{i_E;j_E} \right),$$

where $d_{i_l;j_l}$ are chosen according to (1.9). With these notations the quadratic form $Q$ in (2.2) can be written as

$$Q(U) = \frac{1}{2} (R^r U + d) \cdot (R^r U + d).$$

We now eliminate the boundary conditions (2.6) from the main problem (2.8, 2.9, 2.10). Let $N_b = \sum_{m=1}^M \text{card}(I_m)$. Then the equations (2.6) simply state that the $2N_b$ components of $U$ corresponding to the $N_b$ boundary vertices have prescribed displacements. Without loss of generality assume that the last $2N_b$ components of $U$ correspond to the boundary vertices. Let us partition $U$ as

$$U = \begin{bmatrix} z \\ w \end{bmatrix},$$

where $z = (u^1, u^2, \ldots, u^{N-N_b})$ corresponds to displacement vectors of interior vertices, and $w = (u^{N-N_b+1}, u^{N-N_b+2}, \ldots, u^N)$ corresponds to the displacements of the boundary vertices. The equality constraint (2.6) is now simply

$$w = g,$$

where $g \in \mathbb{R}^{2N_b}$ is the vector of displacements prescribed by the right-hand-sides of (2.6). The matrix $R^r$ can be partitioned similarly to (2.15):

$$R^r = \begin{bmatrix} R \\ R^b \end{bmatrix},$$

where dimensions of $R$ and $R^b$ are $E \times 2(N-N_b)$ and $E \times 2N_b$, respectively. Denote

$$a = R^b g.$$

Using (2.15)–(2.18) in (2.14) and in (2.5) we can reduce the main problem (2.8, 2.9, 2.10) to

**Reduced problem:**

$$\text{minimize } F(z) = \frac{1}{2} (Rz + a + d) \cdot (Rz + a + d)$$

subject to $Rz + a \geq 0$.

The minimization in (2.19) is taken over all $z \in \mathbb{R}^{(N-N_b)}$.

3. **First-order rigidity**

A rigid motion is a composition of a translation and rotation:

$$y(x) = c + x^* + O(x - x^*),$$

where $O$ is an orthogonal (rotation) matrix, $c$ is a translation vector, $x^*$ is a center of rotation. If $O$ is close to identity $I$, then

$$O \approx I + A,$$
where $A$ is a skew matrix ($a_{ij} = -a_{ji}$).

Suppose that in a two-dimensional rigid motion, the rotation angle $\alpha$ is close to zero. Then

\[
O = \begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = I + \alpha K,
\]

where

\[
K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

is a clockwise rotation by $\pi/2$. In that case, (3.1) becomes

\[
(3.2) \quad y(x) = c + x + \alpha K(x - x^*) = \begin{pmatrix} c_1 + x_1^* - \alpha(x - x^*_2) \\ c_2 + x_2^* + \alpha(x - x^*_1) \end{pmatrix}.
\]

Let $u = y(x) - x$ denote the displacement. We can write (3.2) as

\[
(3.3) \quad u(x) = c + \alpha K(x - x^*).
\]

**Definition 3.1.** We call (3.3) an infinitesimal rigid displacements in 2D.

Next, consider a graph. Suppose that a motion of vertices preserves the lengths of the edges. If this implies that the motion is a rigid displacement, then the graph is called rigid. A graph $\Gamma$ is first-order rigid [22] if all solutions of the $R$-system (2.11) are infinitesimally rigid displacements. In addition, $\Gamma$ is independent if the rows of the rigidity matrix $R^r$ are linearly independent. Graphs that are both first-order rigid and independent are called isostatic ([22]). Intuitively, an isostatic graph is minimally rigid, that is removing any edge results in loss of rigidity. Another notion of rigidity is generic rigidity, see [22]. According to thm. 49.1.7 from [22], generic rigidity for a neighborhood in a configuration space is equivalent to the first-order rigidity for some specific configuration in that neighborhood.

With the definition of $R^r$ and $U$ in Sect. 2.3 the system (2.11) can be written as

\[
(3.4) \quad R^r U = 0.
\]

Note the connection between $R$-system and constraints, as well as the functional of the main problem (2.8,2.9,2.10).

The following definition (see [22] for a $d$-dimensional definition) is useful for verifying rigidity of graphs.

**Definition 3.2.** For a graph $\Gamma$, the Henneberg 2-construction in 2D is a sequence of graphs $G_1, G_2, \ldots, G_n$ such that:

(i) $G_{k+1}$ is obtained from $G_k$ by either vertex addition (attaching a new vertex by 2 edges); or edge splitting (replacing and edge from $G_k$ with a new vertex joined to its ends and to 1 other vertex);

(ii) $G_k$ is a complete graph on $k$ vertices, and $G_n = \Gamma$.

The following result is stated in ([22], thm. 49.1.13):

**Theorem 3.3.** If a graph is obtained by a Henneberg $d$-construction, then $G$ is generically isostatic.

In the present case, the rows of rigidity matrix $R^r$ are not linearly independent, but the row rank is maximal. This means that we typically have more edges than needed to ensure rigidity. In this situation, the following theorem (thm. 49.1.14 from [22]) is useful.
Theorem 3.4. If two graphs $G_1$ and $G_2$ are generically rigid planar graphs sharing at least 2 vertices, then the graph $G$ obtained by combining all vertices and edges of $G_1, G_2$ is generically rigid.

4. Existence and uniqueness of minimizers of the reduced problem

Let $\Omega$ be a bounded connected domain in $\mathbb{R}^2$ with a polygonal boundary. First we show that, under certain assumptions on geometry of $\Gamma$, the matrix $R$ has full column rank. We shall say that $\Gamma$ is a triangulation if edges of $\Gamma$ partition $\Omega$ into a disjoint union of triangles.

We also require the following.

Definition 4.1. Consider an interior vertex $x^i$ connected by an edge to a boundary vertex $x^j$. We call $\Gamma$ a regular triangulation if for each such pair $x^i, x^j$ there are boundary vertices $x^k, x^l$, connected to both $x^i$ and $x^j$.

The definition 4.1 states that every edge $(i, j)$ connecting a boundary vertex $j$ with an interior vertex $i$ must be a part of the boundary of two adjacent triangles. These triangles have the property that two of their vertices are boundary vertices $k, l$, and the third vertex is $i$. Physically, this means that the geometry of contacts conforms to the geometry of the boundary, and there are no holes (non-triangular cells) near the boundary.

Proposition 4.2. Suppose that $\Gamma$ is a regular triangulation. Then $\text{rank } R = 2(N - N_b)$.

Proof. Consider a subgraph $\Gamma_{\text{max}} \subseteq \Gamma$ constructed inductively as follows. Begin with $\Gamma_1$ that consists of all boundary vertices. On the next step, add all interior vertices connected to $\Gamma_1$ by two or more non-collinear edges. Also, for each such vertex, add exactly two non-collinear edges that connect this vertex to $\Gamma_1$. Call the resulting graph $\Gamma_2$. Generally, given $\Gamma_k$, $k \geq 2$, define $\Gamma_{k+1} = \Gamma_k \cup S_k$, where $S_k$ consists of all vertices not contained in $\Gamma_k$, connected to $\Gamma_k$ by at least two non-collinear edges, together with pairs of non-collinear edges connecting these vertices to $\Gamma_k$. Since the graph $\Gamma$ has a finite number of vertices, the process terminates after a finite number of steps. The resulting graph is $\Gamma_{\text{max}}$.

We claim that $\Gamma_{\text{max}}$ contains all vertices of $\Gamma$. To show this, consider a graph $\Gamma^c$ which is obtained by adding to $\Gamma_{\text{max}}$ all edges of $\Gamma$ that connect pairs of boundary vertices of $\Gamma_{\text{max}}$. Since $\Gamma$ is a connected triangulation, repeated use of theorem 3.4 yields that $\Gamma$ is first-order rigid. The graph $\Gamma^c$ is a maximal subgraph of $\Gamma$ obtained by the Henneberg construction (see theorem 3.3). Therefore, $\Gamma^c$ is first-order rigid and contains all the vertices of $\Gamma$. Since $\Gamma^c$ and $\Gamma_{\text{max}}$ have the same vertices, the claim is proved.

Next, we claim that the number of edges in $\Gamma_{\text{max}}$ is $2(N - N_b)$. Indeed, each free vertex in $\Gamma_2$ has exactly two non-collinear edges incident at it. Then, on each of the next steps, we add a free vertex together with two non-collinear edges incident at it. Since the number of free vertices in $\Gamma_{\text{max}}$ is $N - N_b$, the claim is proved.

Finally, we claim that the rows of $R$ corresponding to the edges of $\Gamma_{\text{max}}$ are linearly independent. Let the matrix of these rows be denoted by $R_{\text{max}}$. This is a square $2(N - N_b)$ matrix. We claim that an appropriate row-reduction reduces $R_{\text{max}}$ to a matrix $R_{\text{max}}'$ that has block-diagonal form: for each vertex $x^i$ there are exactly two rows $r^{i,1}, r^{i,2}$ of $R_{\text{max}}'$, and two linearly independent unit vectors $q^{i,1}$,
and \(q_{ij}^{12}\), such that \(r_i^{11} (r_i^{12})\) contains \(q_{ij}^{11} (q_{ij}^{12})\) at a place corresponding to \(x_i\) while all other entries in these rows are zero.

To see this, consider first a “basic unit” of \(\Gamma_2\): an interior vertex \(x_i\) and two non-collinear edges incident at it. Let the corresponding unit vectors be \(q_{i1}^{1}, q_{i2}^{1}\). Recall that these edges connect \(x_i\) to two boundary vertices. Consequently, the rows \(r_{1i}, r_{2i}\) corresponding to the above pair of edges have zeros at all places, except two places corresponding to \(x_i\). The non-zero entries of \(r_{1i}, r_{2i}\) are two components of \(q_{i1}^{1}, q_{i2}^{1}\). Since \(q_{i1}^{1}, q_{i2}^{1}\) are linearly independent, so are \(r_{1i}, r_{2i}\). Furthermore, linear combinations of \(r_{1i}, r_{2i}\) can be used to eliminate non-zero entries in other rows. By adding an appropriate linear combination of \(r_{1i}, r_{2i}\) to a row with some unit vector \(q_{ij}^{12}\) at a place corresponding to \(x_i\), we can obtain zeros at this place. Hence, by using \(r_{1i}, r_{2i}\) as pivots in Gaussian elimination we can obtain rows whose only non-zero entries are at a place corresponding to a vertex that was added to \(\Gamma_2\) on the next step of the iteration. These rows, in turn, can be used as pivots. Continuing with row reduction, we can eventually reduce all rows of \(R_{\text{max}}\) to this form (this follows from connectedness of \(\Gamma_{\text{max}}\), which is implied by connectedness of \(\Gamma\)). The proposition is proved.

**Remark.** It is interesting to compare \(R\) and the rigidity matrix \(R^r\). It is well known that for a first-order rigid graph, the null space of \(R^r\) is non-trivial and consists of infinitesimal rigid displacements. The proposition above shows that the null space of \(R\) is trivial. The main difference in structure between these two matrices is that \(R\) contains broken rows. These rows correspond to edges connecting a free vertex to a boundary one. Each broken row has only two non-zero entries which occur at a place corresponding to a free vertex. If a free vertex is connected to two boundary vertices, then the regular triangulation property of \(\Gamma\) ensures that the corresponding broken rows are non-collinear.

**Proposition 4.3.** Consider problem (2.19,2.20). Suppose that \(\Gamma\) is a regular triangulation, the feasible set of (2.19,2.20) is non-empty, and that the unconstrained minimizer of \(F(z)\) is not feasible. Then the problem (2.19,2.20) admits a unique minimizer that is a point on the boundary of its feasible set.

**Proof.** The problem (2.19,2.20) has a feasible point \(\bar{z}\). Then the problem (2.19,2.20) has a unique minimizer \(z^*\) as we now demonstrate. Let the set \(L(\bar{z})\) be defined by

\[
L(\bar{z}) = \{z \in \mathbb{R}^{2(N-N_b)} : F(z) \leq F(\bar{z})\}.
\]

By proposition 4.2 the matrix \(R\) has full column rank. Therefore, the set \(L(\bar{z})\) is an ellipsoid, which is a closed, bounded, convex set. The set \(H\) of \(z \in \mathbb{R}^{2(N-N_b)}\) satisfying the constraint (2.20) is a closed half space. Therefore, \(L(\bar{z}) \cap H\) is a nonempty, closed, bounded convex set. Indeed, the reduced problem (2.19,2.20) is equivalent to the problem

\[
\begin{align*}
\text{(4.1)} & \quad \text{minimize} \quad F(z) \\
\text{(4.2)} & \quad \text{subject to} \quad z \in L(\bar{z}) \cap H.
\end{align*}
\]

Now by the continuity of \(F\) and the compactness of \(L(\bar{z}) \cap H\) we see that the problem (4.1,4.2) and hence the reduced problem (2.19,2.20) has a minimizer \(z^*\). Since \(R\) has full column rank, \(R^T R\) is positive definite. The positive definiteness of the matrix \(R^T R\) implies that \(F\) is strictly convex on \(L(\bar{z}) \cap H\), from which we conclude that \(z^*\) must be unique.
Now if $Rz^* + a > 0$, then $z^*$ must be the unconstrained minimizer of $F$. This contradicts the assumption that the unconstrained minimizer is not feasible. Therefore, some components of $Rz^* + a$ must be zero, and thus $z^*$ must be on the boundary of the feasible region.

5. Optimality conditions for the reduced problem

The reduced problem (2.19, 2.20) has a convex objective function and linear constraints. For such problems it is possible to state optimality conditions that are both necessary and sufficient [17]. To be specific define the Lagrangian $L$ for the problem (2.19, 2.20) by

$$L(z, \lambda) = \frac{1}{2}(Rz + a + d) \cdot (Rz + a + d) - \lambda \cdot (Rz + a).$$

Then $z^*$ solves problem (2.19, 2.20) if and only if there exists $\lambda^*$ such that $z = z^*$ and $\lambda = \lambda^*$ satisfy the Karush, Kuhn, Tucker (KKT) conditions

$$\nabla_z L(z, \lambda) = 0$$
$$Rz + a \geq 0$$
$$\lambda \cdot (Rz + a) = 0$$
$$\lambda \geq 0$$

or equivalently

(5.2) $R^\top (Rz + a + d - \lambda) = 0$
(5.3) $Rz + a \geq 0$
(5.4) $\lambda \cdot (Rz + a) = 0$
(5.5) $\lambda \geq 0$.

See [17, Chapter 12].

Before stating and proving the main result (Theorem 5.1), we list all the assumptions, including both new and previously used.

A1. Consider the problem of minimizing (2.2) subject only to the boundary conditions (2.6, 2.7), but not the constraints (1.2). We assume that the minimizer of that problem is not feasible, that is, this minimizer does not satisfy the impenetrability constraints (1.2).

A2. The network $\Gamma$ is a regular triangulation (as defined in Definition 4.1).

A3. The boundary conditions are prescribed so that

$$|z_i + u_i - (z_j + u_j)| \leq a_i + a_j + \min_{k=1,2,\ldots,N} a_k,$$

for each pair $x_i, x_j$ of boundary vertices in contact.

Let us provide some comments on the nature of assumptions A1–A3. Assumption A1 means that minimizing the energy of the spring network subject only to boundary conditions leads to a configuration in which at least one spring is compressed (and thus violates the impenetrability constraints).

Assumption A2 concerns the contact geometry. The edges of the network split the domain of the problem (a polygon) into elementary cells (triangles). Near the boundary, the cells must be compatible with the geometry of the boundary in the following sense. If a free vertex is connected to a boundary vertex (a corresponding
A particle is in contact with a boundary particle, then it is also in contact with another boundary vertex, located next to the first boundary vertex. Hence, every cell adjacent to the exterior boundary must contain one free vertex and two boundary vertices.

Assumption A3 means that the boundary conditions (2.6, 2.7) are chosen to prevent particles from escaping through the gaps made by displacing the boundary particles. Clearly, if two boundary particles belong to the same group, then no gap can appear between them, and $|\langle x^i + u^i \rangle - \langle x^j + u^j \rangle| = a_i + a_j$. Formation of gaps would be possible between two boundary particles from different groups which are in contact in the reference configuration. If the parameters of rigid body motions in the boundary conditions (2.6, 2.7) are prescribed arbitrarily, then the two particles may move away from each other, and open a gap large enough for a third particle to slip through. Assumption A3 prohibits formation of such gaps.

**Theorem 5.1.** Suppose that assumptions A1-A3 hold. Then there exist $d^* > 0$ and a vector $\delta \in \mathbb{R}^E : \delta_l \in (-1,-1/2), l = 1,2,\ldots,E$, such that for each $d = d\delta$ with $d > 0$, the unique minimizer of (2.19,2.20) has the following property. Each interior vertex $x^i$ of $\Gamma$ has at least two active edges incident at it. The corresponding unit vectors $q^{i,j1}, q^{i,j2}$ must be linearly independent.

**Proof.**

*Step 1.* We claim that A3 implies that there is $c_0 > 0$, which depends on the boundary conditions, but is independent of the choice of $d_{ij}$ in (2.2), such that each feasible displacement $u^i, i = 1,2,\ldots,N$, satisfies

$$|u^i_k| \leq c_0, \quad k = 1,2.$$  

Indeed, first we observe that if $u^i, u^j$ satisfy the linearized constraint (1.2), then they also satisfy the distance constraint (1.1) (the converse is not true in general). Then any feasible collection of displacements also satisfies the distance constraints (1.1) for each pair of neighboring vertices. Now we recall the assumption made in the introduction to conclude that (1.1) must hold for all pairs of vertices. Fix $l \in \{1,2,\ldots,N\}$, corresponding to an interior vertex, and consider a smaller packing $P$ of particles, containing only $D_l$ and all boundary particles. In the reference configuration, $D_l$ is completely surrounded by boundary particles. Then the boundary conditions are prescribed according to A3, the boundary particles still completely confine $D_l$, so that $x^i$ must displace to $x^i + w^i$ that lies inside a certain bounded domain $\Omega'$ that depends only on boundary conditions. Since $x^i_k$ are bounded, this implies that the claim is true for all displacements $w^i$ which are feasible for the smaller packing $P$. Clearly the set of all such displacements is larger than the set of all $u^i$ feasible under all constraints (1.1), and the latter set is larger than the set of all $u^i$ feasible under the linearized constraints (1.2). This proves the claim.

*Step 2.* Let

$$v^i = \sum_{j \in N_i} q^{i,j}, \quad i = 1,2,\ldots,2(N-N_b).$$

First, we prove the theorem under the additional assumption

$$\text{For each } i = 1,2,\ldots,2(N-N_b), \quad v^i \neq s q^{i,j},$$
where $j \in \mathcal{N}_i$, $s \in \mathbb{R}$. Note that (5.9) does not hold for a hexagonal periodic graph, or any periodic graph with an even number of edges incident at a vertex (in this case all $v^i = 0$).

We note that (5.9) implies that

\begin{equation}
|v^i| \geq v_0 > 0
\end{equation}

with $v_0$ independent of $i$. Indeed, $s$ can be zero, so validity of (5.9) means in particular that all $v^i$ are non-zero. Since there is finitely many $v^i$, (5.10) holds.

Consider solutions of the KKT system (5.2, 5.3, 5.4, 5.5). From (5.4), (5.5) it follows that

\begin{equation}
\lambda_j = 0 \text{ if the } j\text{-th constraint is inactive.}
\end{equation}

Let $\theta_j = (Rk + a^j)$. If the $j$-th constraint is active then $\theta_j = 0$, while $\lambda_j$ is arbitrary. Suppose that a feasible point $z^*$ is given. Then $\theta_j$ are given. To solve (5.2) we need to find $\lambda$. Denote by $r^k, k = 1, 2, \ldots, E$ the rows of $R$ (the columns of $R^T$), and suppose that the rows $r^1, r^2, \ldots, r^S$ correspond to the active constraints, and that the rows $r^{S+1}, r^{S+2}, \ldots, r^E$ correspond to the inactive constraints. Choose $d = (-d, -d, \ldots, -d)$. Then (5.2) can be written as

\begin{equation}
-\sum_{s=1}^S r^s \lambda_i + \sum_{j=S+1}^E r^j \theta_l + d \sum_{l=1}^E r^j = 0.
\end{equation}

Pick a vertex $x^i$ of $\Gamma$ and consider the restriction of each $r^j$ in (5.11) to the two components corresponding to $x^i$. Then we have

\begin{equation}
-\sum_{j \in \mathcal{N}_i} \lambda_i q^j + \sum_{j \in \mathcal{N}_i} \theta_i q^j + d v^i = 0,
\end{equation}

where the first sum is taken over active edges incident at $x^i$, while the second sum is over the inactive edges incident at $x^i$.

Next, we determine the minimal number of active edges needed for (5.12) to hold. We can look at (5.12) as a local problem in which $u^i$ may vary, while $u^j, j \in \mathcal{N}_i$ are fixed. Denote by $F_i \subset \mathbb{R}^2$ the feasible region of this local problem. By A3, $F_i$ is a polygon, each side of which corresponds to one or more constraints being active.

In the generic case, one constraint per side is active. In the non-generic case, two or more active constraints correspond to the same side. Since our goal is estimating the number of active constraints from below, it is sufficient to consider only the generic case, corresponding to the “worst case scenario”. In the generic case there are only three possibilities.

**Case 1.** $u^i$ is inside $F_i$. All edges incident at $x^i$ are inactive.

**Case 2.** $u^i$ belongs to only one of the sides of $\partial F_i$. One edge is active.

**Case 3.** $u^i$ is a vertex of $F_i$. Two edges are active.

Consider case 1. Then (5.12) cannot hold for $d$ sufficiently large. Indeed, $|v^i| \geq v_0 > 0$ by assumption, while $|\sum_{j \in \mathcal{N}_i} \theta_i q^j|$ is bounded from above independent of $d$ in view of (5.7).

Consider case 2. Let us number the active edge by $(i, 1)$. Then (5.12) can be written as

\begin{equation}
-\lambda_{i1} q^{i1} + \sum_{j \in \mathcal{N}_i, j > 1} \left((u^i - u^j) \cdot \mathbf{q}^j\right) \mathbf{q}^j + d v^i = 0.
\end{equation}

Enlarging $d$, if necessary, we see that (5.13) can hold only if

\begin{equation}
v^i = s q^{i1},
\end{equation}
where $s < 0$. Since (5.14) is not allowed by (5.8), (5.12) cannot hold for sufficiently large $d$.

Consider case 3. Number the two active edges by $(i, 1)$, $(i, 2)$. The equation (5.12) is

$$
(5.15) \quad -\lambda_{11}q^{11} - \lambda_{12}q^{12} + \sum_{j \in N, j > 2} ((u^i - u^j) \cdot q^{ij}) q^{ij} + dv^i = 0.
$$

For this to hold for large $d$, $v^i$ must be a non-positive linear combination of $q^{11}$, $q^{12}$. These two vectors are linearly independent, otherwise their intersection would not be a vertex of $F_i$. So, Case 3 is possible, provided $v^i$ lies in the negative cone of two active edges.

**Step 3.** Now we remove the assumption (5.9). For each $i = 1, 2, \ldots, (N - N_b)$, and each $\delta \in \mathbb{R}^E$, define a two-dimensional vector $\tilde{v}^i$ to be the restriction of $R^T \delta = \sum_{l=1}^E \delta_l r^i$ to a place $i$. The theorem will be proved is we show that there is a choice of $\delta$ such that $\delta_i \in [1/2, 1]$, and $\tilde{v}^i$ has property (5.9). Indeed, if such $\delta$ is found, we could choose $d = d\delta$, where $d > 0$ is sufficiently large, and repeat the arguments made in the first step, using $\tilde{v}^i$ instead of $v^i$.

To show existence of $\delta$, consider the cube $C_E = \{y \in \mathbb{R}^E : y_l \in (1/2, 1), l = 1, 2, \ldots, E\}$. Pick any point $y^* \in C_E$. Since $C_E$ is open, there is a Euclidean open ball $B(y^*, \rho) \subset C_E$, with the radius $\rho > 0$. Consider the image of $B(y^*, \rho)$ under the mapping $R^T$. Since $R$ has full rank, $R^T$ is surjective, and is therefore an open mapping. Thus, $R^T(B(y^*, \rho))$ contains a Euclidean open ball $B(R^T y^*, \rho^*)$ of a positive radius $\rho^*$ depending only on $R^T$ and $\rho$, but not on $y^*$. If $R^T y^*$ has property (5.9), we choose $\delta = y^*$ and we are done. Otherwise, note that for each $i = 1, 2, \ldots, (N - N_b)$, the ball $B(R^T y^*, \rho^*)$ contains a non-empty two-dimensional Euclidean open ball $B_i$ centered at the restriction of $R^T y^*$ to the place $i$. Since for each $i$ the set $\{v \in \mathbb{R}^2 : v = sq^{ij}, s \in \mathbb{R}, j \in N^i\}$ is a union of a finite number of lines, it cannot contain a two-dimensional ball. Therefore, for each $i = 1, 2, \ldots, (N - N_b)$ there must be a vector $\tilde{v}^i \in B_i$ having property (5.9). Now we can define $\sum_{l=1}^E \delta_l r^i$ via its restrictions $\tilde{v}^i$. Next, by construction, we can find a vector $\delta \in B(y^*, \rho) \subset C_E$ such that $R^T \delta = \sum_{l=1}^E \delta_l r^i$.

The theorem is proved.

**6. Order parameter**

Recently, a phenomenological theory of slow dense granular flows was proposed in [1, 21]. A key quantity in that theory is the order parameter, defined as the ratio of the number of solid-like contacts to the number of all contacts within a given control volume. In [21], a contact is considered solid-like if two particles are jammed together for longer than a characteristic collision time. The relevant characteristic time is $\tau = a/v_a$, where $a$ is particle radius and $v_a$ is the speed of sound in a solid material of the particles. Our model corresponds to the instantaneous material response, when $\tau$ is much smaller than other relaxation times in the system, such as the ratio of the sample size to a typical particle velocity.

An obvious type of pair motion leading to a solid-like contact is a rigid displacement (a pair of particles infinitesimally moves as a rigid body). We shall call this type of contacts stuck. If a contact between $D_i$ and $D_j$ is stuck, then
\((u^i - u^j) \cdot q^{ij} = 0\), which is easy to check using the definition of rigid displacements. This means that the impenetrability constraint for the corresponding edge of \((i, j)\) of the network is satisfied as an equation (the edge is active). However, not every active edge corresponds to a stuck contact. Another type of a local motion that produces \((u^i - u^j) \cdot q^{ij} = 0\) is an infinitesimal shear motion when \(u^i - u^j\) is orthogonal to \(q^{ij}\). The corresponding contact is called sheared. Note also that infinitesimal shear is the same as infinitesimal rotation, so this type of motion includes infinitesimal rolling as well as shear sliding.

We consider both sheared and stuck contact as solid-like, because stuck contacts are stable, while sheared contacts in an actual granular material will be subject to friction. Friction can be viewed as partially stabilizing, at least when the shearing force is below the static friction threshold. Such non-sliding frictional contacts are considered as solid-like in the simulations performed in [21]. In addition, some heuristic arguments and numerical simulations presented in [7, 8], suggest that friction enhances elastic behavior of sufficiently large samples. Therefore, it makes sense to think of the network of solid-like contacts as the main load-bearing structure and call this network strong. In contrast, a broken contact satisfying (1.3) corresponds to a local weakening in the material because in this case two particles separate completely. We can think of the network of all broken contacts as weak. Moreover, division of contacts into broken and solid-like corresponds to the division of constraints into active and inactive, as done in optimization theory. Therefore, this division is natural mathematically, and also makes sense from the physics point of view.

In addition, the definition in [21] does not sufficiently clarify the nature of averaging. The notion of an order parameter in static problems should not use time averaging. The result of spatial averaging depends on the size of the sample that is being averaged. Thus, if the order parameter is obtained by, say, spatial averaging, then it must depend on both location and size of the “control volume”. In the discrete situation, the size of the averaging sample can be measured by the minimal number of edges connecting a pair of vertices within the sample.

This suggests a definition of the size-dependent order parameter. To state this definition we first define the averaging sample.

**Definition 6.1.** A vertex \(x^i\) is in the \(k\)-th neighborhood of \(x^j\) if \(\Gamma\) contains a path connecting \(x^i\) and \(x^j\) with no more than \(k\) edges.

Now, to each \(k\)-neighborhood we can associate a value of an order parameter.

**Definition 6.2.** For each \(x^i\) and each non-negative integer \(k \leq N\), the size dependent order parameter \(\rho(x^i, k)\) is defined by

\[
\rho(x^i, k) = \frac{\sum_k n_s}{\sum_k n},
\]

where the numerator is the number of active edges in \(k\)-neighborhood of \(x^i\), and denominator is the number of all edges in that neighborhood.

Theorem 5.1 implies the lower bound

\[
\rho(x^i, N) \geq \frac{N}{E}
\]

on the order parameter associated with the maximal, \(N\)-th neighborhood of each interior vertex \(x^i\). Indeed, counting active edges (two per vertex) gives \(2N\) edges,
each counted at most twice. In particular, (6.2) means that the order parameter \( \rho(x^i, N) \) is bounded from below by the reciprocal of the mean coordination number of the network.

7. Conclusions

We have studied a linear spring network model of granular statics without friction. In our model, the packing was represented by a network of linear elastic springs. Each spring corresponds to a contact between two particles. Geometric impenetrability constraints within the packing were modeled by the linearized impenetrability constraints on the displacements of the vertices of the network. The constraints have the form of linear inequalities, that can be satisfied either as an equality (an active constraint), or as a strict inequality (inactive constraint). Constraints are in one-to one correspondence with the edges of the network, which makes it natural to consider active and inactive edges. The question addressed in the paper was to estimate the number and distribution of active edges in the energy-minimizing configuration. We showed that each interior vertex of the network has at least two active edges incident at it. This result qualitatively reproduces the micro-band structure obtained in [14, 15] by numerical simulations. We also discussed the connection between our result and a lower bound on the order parameter [1, 21]. In the paper, we proposed a definition of the order parameter that is similar to the one introduced in [21], but differs from it in the interpretation of the so-called solid-like contacts. On the one hand, our definition appears to be in accord with a physical picture of granular statics, recently proposed in [7, 8]. On the other hand, it is a naturally related to optimization theory.

References


Department of Mathematics, Washington State University, Pullman, WA, 99164
E-mail address: ari@math.wsu.edu

Department of Mathematics and Materials Research Institute, Penn State University, University Park, PA 16802, USA
E-mail address: berlyand@math.psu.edu

Department of Mathematics, Washington State University, Pullman, WA, 99164
E-mail address: panchenko@math.wsu.edu