Coherent systems of components with multivariate phase type life distributions

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Abstract

This paper discusses engineering systems with structural as well as failure dependence among their components. Using the Markov process approach, a coherent system of components with multivariate phase type life distribution is introduced to model the components working in a common random environment. The phase type model is flexible enough to comprise many realistic situations, and still manages to yield the explicit matrix representations of the system reliability measures. Some examples are also presented to illustrate the fact that ignoring the failure dependence among the components often results in over-estimating or under-estimating the system reliability.

Key words and phrases: Coherent system, $k$-out-of-$n$ system, system of dependent components, multivariate phase type life distribution, Marshall-Olkin distribution.

1 Introduction

Many engineering systems experience structural as well as failure dependence among their components. Most studies, however, on the coherent systems have been focused on struc-
tural dependence. The $k$-out-of-$n$ system, consecutive $k$-out-of-$n$ system, and other popular coherent structures are extensively studied only for the components that are independent. The focus of this paper is on the coherent systems of stochastically dependent components.

The failure dependence arises due to common production and operating environments, and such a common cause failure often poses serious threats to the reliability of many operating systems. Ignoring the failure dependence among the system components often results in over-estimating or under-estimating the system reliability. Of course, by properly modeling, it is sometimes possible to preclude dependency among the components. But it is necessary in many cases to establish a model that explicitly takes into account the dependency (Aven and Jensen 1999). The system reliability assessment for dependent components has become one of the most difficult tasks in reliability engineering.

It is well-known that the reliability of a $k$-out-of-$n$ system or consecutive $k$-out-of-$n$ system of independent components can be calculated explicitly from the component reliabilities (see, for example, Chang, Cui and Hwang 2000). The coherent system was introduced by Birnbaum, Esary and Saunders (1961), and some latest development was presented in Dutuit and Rauzy (2001), and Levitin (2001), and the references therein. However, this black-box approach is inadequate for the systems of dependent components, because the system reliability for dependent components depends not only on (marginal) component reliabilities but also on the dependence structure among the components. To overcome this difficulty, we employ the Markov process approach. A Markov process can be used to describe the aging behaviors of components under the common environmental factors that induce the failure dependence, and by properly defining failure states, its first passage times to these failure states can be used to model the lifetimes of various coherent systems. In other words, the evolution of a Markov process captures the failure dependence, and the structure of its state space describes the structural dependence of the components.

This Markovian method leads naturally to the multivariate phase type distributions, introduced by Assaf et al. (1984). Thus, the component lifetimes to be studied in this paper are modeled according to a multivariate phase type life distribution. It is well-known that the phase type distributions are tractable and can be used to approximate any life distributions. Our phase type model is flexible enough to comprise many realistic situations, and still manages to yield the matrix representations of the system reliability measures.

The paper is organized as follows. Section 2 discusses the multivariate phase type distri-
butions, and obtains explicitly the matrix representations of the system reliability and mean time to failure for a coherent system of components that have a multivariate phase type distribution. Section 3 focuses on three popular coherent structures with multivariate phase type component lifetimes: $k$-out-of-$n$, consecutive $k$-out-of-$n$, and $r$-within-consecutive $k$-out-of-$n$ systems. Section 4 presents an example using the Marshall-Olkin distribution to illustrate our results.

2 Coherent Systems and Multivariate Phase Type Distributions

Consider a system comprising $n$ components. We assign a binary variable $x_i$ to component $i$:

\[
x_i = \begin{cases} 
1 & \text{if component } i \text{ is in the functioning state} \\
0 & \text{if component } i \text{ is in the failure state.}
\end{cases}
\]

Similarly, the binary variable $\Phi$ indicates the state of the system:

\[
\Phi = \begin{cases} 
1 & \text{if the system is in the functioning state} \\
0 & \text{if the system is in the failure state.}
\end{cases}
\]

We assume that

\[
\Phi = \Phi(\mathbf{x}),
\]

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, that is, the state of the system is determined completely by the states of the components. We refer to the function $\Phi(\mathbf{x})$ as the structure function of the system. Let $(0_i, \mathbf{x})$ denote the state vector $\mathbf{x}$ emphasizing that the state of its $i$-th component is 0, and $(1_i, \mathbf{x})$ denote the state vector $\mathbf{x}$ emphasizing that the state of its $i$-th component is 1. A system is said to be coherent (see, for example, Barlow and Proschan 1981) if

1. the structure function $\Phi$ is nondecreasing in each argument, and

2. each component is relevant, i.e., there exists at least one vector $\mathbf{x}$ such that $\Phi(1_i, \mathbf{x}) = 1$ and $\Phi(0_i, \mathbf{x}) = 0$.

The coherent system covers many realistic engineering systems. For example, series and parallel systems are coherent. More generally, a $k$-out-of-$n$ system is coherent.
Example 2.1. A system that fails if and only if at least \( k \) out of \( n \) components fail is called a \( k \)-out-of-\( n \): \( F \) system. The structure function for this system is given by

\[
\phi(x) = \begin{cases} 
1 & \text{if } \sum_{i=1}^{m} x_i > n - k \\
0 & \text{if } \sum_{i=1}^{m} x_i \leq n - k.
\end{cases}
\]

A system that fails if and only if at least \( k \) consecutive components fail is called a consecutive \( k \)-out-of-\( n \): \( F \) system. The structure function for this system is given by

\[
\phi(x) = \begin{cases} 
1 & \text{if } \sum_{i=l+1}^{l+k} x_i > 0 \ 	ext{for any } l, \\
0 & \text{otherwise}.
\end{cases}
\]

Clearly, these systems are coherent systems.

Consider a system of \( n \) components with the coherent structure function \( \Phi(x) \). Let \( T_i \) denote the lifetime of component \( i \), \( 1 \leq i \leq n \). Define

\[ X_i(t) = I_{(T_i>t)}, \ i = 1, \ldots, n, \]

where \( I_A \) denotes the indicator function of the set \( A \). Observe that the \( i \)-th component is functional at time \( t \) if and only if \( X_i(t) = 1 \). Thus, the system works at time \( t \) if and only if \( \Phi(X_1(t), \ldots, X_n(t)) = 1 \), and the system reliability at time \( t \) is given by

\[ R_\Phi(t) = \Pr\{\Phi(X_1(t), \ldots, X_n(t)) = 1\}. \]

Let \( \tau_\Phi(T_1, \ldots, T_n) \) denote the lifetime of the coherent system \( \Phi \) of the components with lifetimes \( T_1, \ldots, T_n \), where \( \tau_\Phi \) is called in the literature the coherent life function associated with the coherent structure function \( \Phi \). The system reliability at time \( t \) can be also written as

\[ R_\Phi(t) = \Pr\{\Phi(X_1(t), \ldots, X_n(t)) = 1\} = \Pr\{\tau_\Phi(T_1, \ldots, T_n) > t\}. \quad (2.1) \]

The system mean time to failure (MTTF) is given by \( E(\tau_\Phi(T_1, \ldots, T_n)) \).

In many applications, the component lifetimes \( T_1, \ldots, T_n \) are stochastically dependent, and as such, are distributed according to some multivariate life distribution. For a coherent system with independent components, its reliability at time \( t \) is completely determined by the system structure and component reliabilities at time \( t \). But this is not the case for the dependent components. Unlike the independent case, the aging process of a coherent system of dependent components depends not only on the component aging behavior and system
structure, but also on the component correlation. As we illustrate as follows, the reliability assessment for a coherent system of dependent components needs to take into account the past component aging evolution histories.

Let \( X(t), t \geq 0 \) be a right-continuous, continuous-time Markov chain on a finite system state space \( \mathcal{E} \) with generator \( Q \), where \( X(t) \) describes the system evolution at time \( t \). Let \( \mathcal{E}_i, i = 1, \ldots, n \), be \( n \) nonempty stochastically closed subsets of \( \mathcal{E} \) such that \( \cap_{i=1}^n \mathcal{E}_i \) is a proper subset of \( \mathcal{E} \) (a subset of the state space is said to be stochastically closed if once the process \( \{X(t)\} \) enters it, \( \{X(t)\} \) never leaves). Intuitively, subset \( \mathcal{E}_i \) describes the failure states for component \( i, i = 1, \ldots, n \). We assume that absorption into \( \cap_{i=1}^n \mathcal{E}_i \) is certain. Since we are interested in the process only until it is absorbed into \( \cap_{i=1}^n \mathcal{E}_i \), we may assume, without loss of generality, that \( \cap_{i=1}^n \mathcal{E}_i \) consists of one absorbing state, which we shall denote by \( \Delta \). Thus, without loss of generality, we may write \( \mathcal{E} = (\cap_{i=1}^n \mathcal{E}_i) \cup \mathcal{E}_0 \) for some subset \( \mathcal{E}_0 \subset \mathcal{E} \) with \( \mathcal{E}_0 \cap \mathcal{E}_j = \emptyset \) for \( 1 \leq j \leq n \). The system states in \( \mathcal{E} \) are enumerated in such a way that \( \Delta \) is the last element of \( \mathcal{E} \). Thus, the generator of the system evolution process has the form

\[
Q = \begin{bmatrix}
A & -Ae \\
0 & 0
\end{bmatrix},
\]

where \( 0 = (0, \ldots, 0) \) is the row vector of zeros, \( e = (1, \ldots, 1)^T \) is the column vector of 1’s, sub-generator \( A \) is a \((|\mathcal{E}| - 1) \times (|\mathcal{E}| - 1)\) nonsingular matrix. Let \( \beta = (\alpha, 0) \) be an initial probability vector on \( \mathcal{E} \), and so the system evolution starts within \( \mathcal{E} - \{\Delta\} \) almost surely.

The lifetime of a component is the time that the system evolution process \( \{X(t)\} \) first enters its failure state set. Thus, we define the lifetime of the \( i \)-th component as

\[
T_i = \inf\{ t \geq 0 : X(t) \in \mathcal{E}_i \}, \quad i = 1, \ldots, n.
\]

As in Assaf et al. (1984), for simplicity, we shall assume that \( \Pr\{T_1 > 0, \ldots, T_n > 0\} = 1 \), which means that the underlying Markov chain \( \{X(t)\} \) starts within \( \mathcal{E}_0 \) almost surely. The component lifetime vector \( (T_1, \ldots, T_n) \) is called a phase type random vector, and its distribution is called a multivariate phase type distribution (MPH) with representation \((\alpha, A, \mathcal{E}, \mathcal{E}_1, \ldots, \mathcal{E}_n)\). Thus a phase type random vector is a vector of correlated first-passage times of \( \{X(t)\} \) to various overlapping subsets of the state space \( \mathcal{E} \).

Note that \( \mathcal{E}_i \)'s are all stochastically closed. Thus, once the process \( \{X(t)\} \) enters some \( \mathcal{E}_i \)'s, it can not move out and will move further into more \( \mathcal{E}_i \)'s. Eventually, the system evolution process \( \{X(t)\} \) will be absorbed into \( \Delta \), and when this happens, all the components fail. For
example, the process \( \{X(t)\} \) can enter \( E_1 \) first, and then \( E_2, \ldots \), and finally enters \( E_n \), thus we have the situation that \( T_1 < T_2 \cdots < T_n \). Of course, it is also possible that the process \( \{X(t)\} \) enters several \( E_i \)'s at the same time, which corresponds to the situation that several components fail simultaneously. The multivariate phase type distribution indeed models a wide variety of situations.

The marginal distribution of \( T_j, 1 \leq j \leq n \), with representation \((\alpha, A, E, E_j)\) reduces to the univariate phase type distribution (PH) introduced in Neuts (1981). In this case, \( T_j \) is the first passage time of the Markov chain \( \{X(t)\} \) that reaches the subset \( E_j \). We partition the sub-generator \( A \) in (2.2) as follows.

\[
A = \begin{bmatrix}
A_{E_i-E_j} & A_{E_i-E_j, E_j} \\
A_{E_j, E_i-E_j} & A_{E_j}
\end{bmatrix},
\]

(2.4)

where \( A_{E_i-E_j} \) (\( A_{E_j} \)) is the sub-matrix of all the transition rates within the subset \( E - E_j \) (\( E_j \)). Since \( E_j \) is stochastically closed, then \( A_{E_j, E_i-E_j} = 0 \). Correspondingly, the vector \( \alpha \) is partitioned as

\[
\alpha = (\alpha_{E_i-E_j}, \alpha_{E_j}).
\]

It is known from Neuts (1981) that

\[
\Pr(T_j > t) = \frac{\alpha_{E_i-E_j}}{\alpha_{E_i-E_j}} e^{A_{E_i-E_j}t} e,
\]

(2.5)

\[
E(T_j) = -\frac{\alpha_{E_i-E_j}}{\alpha_{E_i-E_j}} A_{E_i-E_j}^{-1} e.
\]

(2.6)

The Laplace transforms and higher moments can be also calculated explicitly.

In general, phase type distributed random variables \( T_1, \ldots, T_n \) are dependent. Li (2003) obtained a sufficient condition for \((T_1, \ldots, T_n)\) to be positively correlated. Some examples presented in Freund (1961) show that a phase type random vector arising from completing resource problems can be also negatively correlated. Any independent, phase type distributed random variables clearly have a multivariate phase type distribution, and so a coherent system with independent, phase type distributed components is a special case of our multivariate phase type models.

Examples of MPH distributions include, among many others, the well-known Marshall-Olkin distribution (Marshall and Olkin 1967). The MPH distributions, their properties, and some related applications in reliability theory were discussed in Assaf et al. (1984). As in
the univariate case, those MPH distributions (and their densities, Laplace transforms and moments) can be written in a closed form. The set of \( n \)-dimensional MPH distributions is dense in the set of all life distributions on \([0, \infty)^n\). It is also stated in Assaf et al. (1984) that MPH distributions is closed under the formation of coherent systems. To illustrate the versatility of the phase type approach, we provide below a short proof of this result.

**Theorem 2.2.** Let \( T = (T_1, \ldots, T_n) \) be an \( n \)-dimensional MPH random vector with representation \((\alpha, A, \mathcal{E}, \mathcal{E}_1, \ldots, \mathcal{E}_n)\), and \( \tau_1, \ldots, \tau_m \) be \( n \)-variate coherent life functions associated with the \( m \) coherent structure functions \( \Phi_1, \ldots, \Phi_m \) with \( \Phi_j \neq 0, \ j = 1, \ldots, m \). Then \((\tau_1(T), \ldots, \tau_m(T))\) is an \( m \)-dimensional MPH random vector with representation \((\alpha, A, \mathcal{E}, \Lambda_1, \ldots, \Lambda_m)\), where

\[
\Lambda_i = \{ e \in \mathcal{E} : \Phi_i(I_{\mathcal{E}_i^c}(e), \ldots, I_{\mathcal{E}_n^c}(e)) = 0 \}, \ i = 1, \ldots, m,
\]

and \( \mathcal{E}_i^c \) denotes the complement of \( \mathcal{E}_i \).

**Proof.** Let \( \{X(t)\} \) be a Markov system evolution process that defines \( T \). Observe that for \( i = 1, \ldots, n \), the \( i \)-th component is functional at time \( t \) if and only if \( X(t) \in \mathcal{E}_i \).

For each \( 1 \leq j \leq m \), the system lifetime \( \tau_j(T) \) is the first passage time that \( \{X(t)\} \) reaches one of its failure state \( e \in \mathcal{E} \) with

\[
\Phi_j(I_{\mathcal{E}_i^c}(e), \ldots, I_{\mathcal{E}_n^c}(e)) = 0.
\]

Thus, we have

\[
\tau_j(T) = \inf\{t : X(t) \in \Lambda_j\}, \ j = 1, \ldots, m.
\]

Since \( \mathcal{E}_i \)'s are all stochastically closed, then we have \( I_{\mathcal{E}_i^c}(X(t)) = 0 \) if and only if \( I_{\mathcal{E}_i^c}(X(t')) = 0 \) for any \( t' \geq t \). Since any coherent function is nondecreasing, we obtain that for any \( t' \geq t \), if \( \Phi_i(I_{\mathcal{E}_i^c}(X(t)), \ldots, I_{\mathcal{E}_n^c}(X(t))) = 0 \), then

\[
\Phi_i(I_{\mathcal{E}_i^c}(X(t')), \ldots, I_{\mathcal{E}_n^c}(X(t'))) \leq \Phi_i(I_{\mathcal{E}_i^c}(X(t)), \ldots, I_{\mathcal{E}_n^c}(X(t))) = 0.
\]

Thus \( \Lambda_j \)'s are all stochastically closed. \( \square \)

As illustrated in Theorem 2.2, by properly defining the sets of failure states, our phase type distributions can model any coherent system with dependent components. Theorem 2.2 can be used to study the reliability of a network of sub-systems, and in particular, yields the following formulas for the coherent systems with phase type distributed components.
**Corollary 2.3.** Let \( \mathbf{T} = (T_1, \ldots, T_n) \) be an \( n \)-dimensional MPH random vector with representation \((\alpha, A, \mathcal{E}, \mathcal{E}_1, \ldots, \mathcal{E}_n)\), and \( \tau_\Phi \) be an \( n \)-variate coherent life function associated with the coherent structure function \( \Phi \) with \( \Phi \neq 0 \).

1. \( \tau_\Phi(\mathbf{T}) \) is of phase type with representation \((\alpha, A, \mathcal{E}, \Lambda)\), where
   \[
   \Lambda = \{e \in \mathcal{E} : \Phi(I_{E_1}(e), \ldots, I_{E_n}(e)) = 0\}.
   \]

2. The reliability of the coherent system \( \Phi \) at time \( t \) is given by
   \[
   R_\Phi(t) = \frac{\alpha_{\mathcal{E}-\Lambda}}{\alpha_{\mathcal{E}-\Lambda} e^{A_{\mathcal{E}-\Lambda} t}} e.
   \]

3. The system mean time to failure is given by
   \[
   E(\tau_\Phi(\mathbf{T})) = -\frac{\alpha_{\mathcal{E}-\Lambda}}{\alpha_{\mathcal{E}-\Lambda} e^{A_{\mathcal{E}-\Lambda}^{-1}}} e.
   \]

### 3 Reliability Structures of Dependent Components

In this section, we use Corollary 2.3 to obtain the formulas of the system reliability and MTTF for the \( k \)-out-of-\( n \): F system, consecutive \( k \)-out-of-\( n \): F system, and \( r \)-within-\( \cdot \)Consecutive \( k \)-out-of-\( n \): F system. We only discuss the F systems, and the formulas for the G systems can be similarly obtained. As we mention earlier, the system evolution process \( \{X(t)\} \) describes the failure dependence among the components. In each of the following systems, we now demonstrate that various structural dependencies can be established by proper constructions of failure state subsets \( \Lambda \). The same technique can be applied to any coherent system.

#### 3.1 \( k \)-out-of-\( n \): F systems

Consider a \( k \)-out-of-\( n \): F system \( \Phi \) with components whose joint life distribution is of phase type with representation \((\alpha, A, \mathcal{E}, \mathcal{E}_1, \ldots, \mathcal{E}_n)\). Define

\[
\mathcal{O}_k = \bigcup \{ (i_1, i_2, \ldots, i_k) \subseteq \{1, 2, \ldots, n\} \} \cap \bigcap_{j=1}^k E_{i_j}, \ 1 \leq k \leq n.
\]

For example, consider the case when \( n = 3 \), where the state space

\[
\mathcal{E} = \{0, 1, 2, 3, 12, 13, 23, 123\}
\]
with the absorbing state $\Delta = 123$ and stochastically closed subsets

\[ \mathcal{E}_1 = \{1, 12, 13, 123\}, \quad \mathcal{E}_2 = \{2, 12, 23, 123\}, \quad \mathcal{E}_3 = \{3, 13, 23, 123\}. \quad (3.3) \]

Then we have

\[ \mathcal{O}_1 = \bigcup_{i=1}^{3} \mathcal{E}_i = \{1, 2, 3, 12, 13, 23, 123\}, \]
\[ \mathcal{O}_2 = (\mathcal{E}_1 \cap \mathcal{E}_2) \cup (\mathcal{E}_1 \cap \mathcal{E}_3) \cup (\mathcal{E}_2 \cap \mathcal{E}_3) = \{12, 13, 23, 123\}, \]
\[ \mathcal{O}_3 = \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3 = \{123\}. \]

It follows from Corollary 2.3 that the $k$-out-of-$n$: F system lifetime has a univariate phase type distribution with representation $(\alpha, A, \mathcal{O}_k)$. Thus, the $k$-out-of-$n$: F system reliability at time $t$ is given by

\[ R_{(k,n)}(t) = \frac{\alpha_{\mathcal{O}_k} - \alpha_{\mathcal{E}_0}}{\alpha_{\mathcal{E}_0}} e^{A(\mathcal{E}_0) - \mathcal{O}_k} e, \quad (3.4) \]

and its mean time to failure is given by

\[ \text{MTTF}_{(k,n)} = - \frac{\alpha_{\mathcal{O}_k} - \alpha_{\mathcal{E}_0}}{\alpha_{\mathcal{E}_0}} A_{\mathcal{E}_0}^{-1} \mathcal{E}_{\mathcal{O}_k}. \quad (3.5) \]

### 3.2 Consecutive $k$-out-of-$n$: F systems

Consider a consecutive $k$-out-of-$n$: F system $\Phi$ with components whose joint life distribution is of phase type with representation $(\alpha, A, \mathcal{E}, \mathcal{C}_k)$. Define

\[ \mathcal{C}_k = \bigcup_{l=0}^{n-k} (\cap_{j=1}^{k} \mathcal{E}_{l+j}), \quad 1 \leq k \leq n. \quad (3.6) \]

For the state space given in (3.2) and (3.3), we have

\[ \mathcal{C}_2 = (\mathcal{E}_1 \cap \mathcal{E}_2) \cup (\mathcal{E}_2 \cap \mathcal{E}_3) = \{12, 23, 123\}. \]

Then the consecutive $k$-out-of-$n$: F system lifetime has a univariate phase type distribution with representation $(\alpha, A, \mathcal{C}_k)$. Thus, the consecutive $k$-out-of-$n$: F system reliability at time $t$ is given by

\[ R_{C(k,n)}(t) = \frac{\alpha_{\mathcal{C}_k} - \alpha_{\mathcal{E}_0}}{\alpha_{\mathcal{E}_0}} e^{A(\mathcal{E}_0) - \mathcal{C}_k} e, \quad (3.7) \]

and its mean time to failure is given by

\[ \text{MTTF}_{C(k,n)} = - \frac{\alpha_{\mathcal{C}_k} - \alpha_{\mathcal{E}_0}}{\alpha_{\mathcal{E}_0}} A_{\mathcal{E}_0}^{-1} \mathcal{E}_{\mathcal{C}_k}. \quad (3.8) \]
3.3 \( r\)-within-Consecutive \( k\)-out-of-\( n\): F systems

An \( r\)-within-consecutive \( k\)-out-of-\( n\): F system consists of \( n\) linearly ordered components such that the system fails if and only if there exists \( k\) consecutive components which include among them at least \( r\) failed components. Consider an \( r\)-within-consecutive \( k\)-out-of-\( n\): F system \( \Phi \) with components whose joint life distribution is of phase type with representation \((\alpha, A, \mathcal{E}, \mathcal{E}_1, \ldots, \mathcal{E}_n)\). Define

\[
\mathcal{R}_{(r,k,n)} = \bigcup_{l=0}^{n-k} \left( \bigcup_{\{i_1, \ldots, i_r\} \subseteq \{l+1, \ldots, l+k\}} \left( \bigcap_{j=1}^{r} \mathcal{E}_{i_j} \right) \right), \quad 1 \leq k \leq n. \tag{3.9}
\]

For example, consider a \( 2\)-within-consecutive-3-out-of-4: F system of the components whose underlying Markov chain has the state space

\[\mathcal{E} = \{0, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 134, 234, 1234\},\]

and the stochastically closed subsets,

\[
\mathcal{E}_1 = \{12, 13, 14, 123, 124, 134, 1234\},
\]

\[
\mathcal{E}_2 = \{12, 23, 24, 123, 124, 234, 1234\},
\]

\[
\mathcal{E}_3 = \{13, 23, 34, 123, 134, 234, 1234\},
\]

\[
\mathcal{E}_4 = \{14, 24, 34, 124, 134, 234, 1234\}.
\]

Thus, \( \mathcal{R}_{(2,3,4)} = \{12, 13, 23, 24, 34, 123, 234\} \). The \( r\)-within-consecutive \( k\)-out-of-\( n\): F system lifetime has a univariate phase type distribution with representation \((\alpha, A, \mathcal{E}, \mathcal{R}_{(r,k,n)})\). Thus, the \( r\)-within-consecutive \( k\)-out-of-\( n\): F system reliability at time \( t \) is given by

\[
R_{RC(r,k,n)}(t) = \frac{\alpha_{\mathcal{E} - \mathcal{R}_{(r,k,n)}}}{\alpha_{\mathcal{E} - \mathcal{R}_{(r,k,n)}}} e^{A_{\mathcal{E} - \mathcal{R}_{(r,k,n)}}^t} e, \tag{3.10}
\]

and its mean time to failure is given by

\[
MTTF_{RC(r,k,n)} = -\frac{\alpha_{\mathcal{E} - \mathcal{R}_{(r,k,n)}}}{\alpha_{\mathcal{E} - \mathcal{R}_{(r,k,n)}}} A_{\mathcal{E} - \mathcal{R}_{(r,k,n)}}^{-1} e. \tag{3.11}
\]
4 Systems of Components with Marshall-Olkin Life Distributions

Let \( \{E_S, S \subseteq \{1, \ldots, n\}\} \) be a sequence of independent, exponentially distributed random variables, with \( E_S \) having mean \( 1/\lambda_S \). Let

\[
T_j = \min\{E_S : S \ni j\}, \quad j = 1, \ldots, n.
\] (4.1)

The joint distribution of \((T_1, \ldots, T_n)\) is called the Marshall-Olkin distribution with parameters \( \{\lambda_S, S \subseteq \{1, \ldots, n\}\} \) (Marshall and Olkin 1967). In the reliability context, \( T_1, \ldots, T_n \) can be viewed as the lifetimes of \( n \) components operating in a random shock environment where a fatal shock governed by Poisson process \( \{N_S(t), t \geq 0\} \) with rate \( \lambda_S \) destroys all the components with indexes in \( S \subseteq \{1, \ldots, n\} \) simultaneously. Assume that these Poisson shock arrival processes are independent, then,

\[
T_j = \inf\{t : N_S(t) \geq 1, S \ni j\}, \quad j = 1, \ldots, n.
\] (4.2)

Let \( \{M_S(t), t \geq 0\}, S \subseteq \{1, \ldots, n\} \), be independent Markov chains with absorbing state \( \Delta_S \), each representing the exponential distribution with parameter \( \lambda_S \). It follows from (4.2) that \((T_1, \ldots, T_n)\) is of phase type with the underlying Markov chain on the product space of these independent Markov chains with absorbing classes \( \mathcal{E}_j = \{(e_S) : e_S = \Delta_S \text{ for some } S \ni j\}, \ 1 \leq j \leq n \).

It is easy to verify that the marginal distribution of the \( j \)-th component of the Marshall-Olkin distributed random vector is exponential with mean \( 1/\sum_{S : S \ni j} \lambda_S \).

To calculate the reliabilities of the coherent systems, we need to simplify the underlying Markov chain for the Marshall-Olkin distribution and obtain its phase type representation.

Let \( \{X(t), t \geq 0\} \) be a Markov chain with state space \( \mathcal{E} = \{S : S \subseteq \{1, \ldots, n\}\} \), and starting at \( \emptyset \) almost surely. The index set \( \{1, \ldots, n\} \) is the absorbing state, and

\[
\mathcal{E}_0 = \{\emptyset\}
\]

\[
\mathcal{E}_j = \{S : S \ni j\}, \quad j = 1, \ldots, n.
\]

It follows from (4.2) that its sub-generator is given by \( A = (a_{e,e'}) \), where

\[
a_{e,e'} = \sum_{L : L \subseteq S', L \cup S = S'} \lambda_L, \quad \text{if } e = S, \ e' = S' \text{ and } S \subset S',
\]
\[ a_{e,e} = \sum_{L: L \subseteq S} \lambda_L - \Lambda, \text{ if } e = S \text{ and } \Lambda = \sum_{S} \lambda_S, \]

and zero otherwise. Using the results in Section 3 and these parameters, we can calculate the reliabilities. To illustrate the results, we consider the three dimensional case that \( n = 3 \).

To simplify the notation, we express a subset \( S \) by listing its elements. For example, \( 0 \) denotes subset \( \emptyset \), and \( 12 \) denotes subset \( \{1, 2\} \). Thus, the state space for a three dimensional Marshall-Olkin distribution is \( \mathcal{E} = \{0, 1, 2, 3, 12, 13, 23, 123\} \) with

\[ \mathcal{E}_1 = \{1, 12, 13, 123\}, \]
\[ \mathcal{E}_2 = \{2, 12, 23, 123\}, \]
\[ \mathcal{E}_3 = \{3, 13, 23, 123\}, \]

where 123 is the absorbing state. The initial probability vector is \((1, 0, 0, 0, 0, 0, 0, 0)\), and its sub-generator \( A \) is given by

\[
\begin{bmatrix}
-\Lambda & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_{12} & \lambda_{13} & \lambda_{23} \\
0 & -\Lambda - \lambda_1 & 0 & 0 & \lambda_2 + \lambda_{12} & \lambda_3 + \lambda_{13} & 0 \\
0 & 0 & -\Lambda - \lambda_2 & 0 & \lambda_1 + \lambda_{12} & 0 & \lambda_3 + \lambda_{23} \\
0 & 0 & 0 & -\Lambda - \lambda_3 & 0 & \lambda_1 + \lambda_{13} & \lambda_2 + \lambda_{23} \\
0 & 0 & 0 & 0 & -\Lambda - \sum_{J \subseteq \{1,2\}} \lambda_J & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\Lambda - \sum_{J \subseteq \{1,3\}} \lambda_J & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\Lambda - \sum_{J \subseteq \{2,3\}} \lambda_J
\end{bmatrix},
\]

where \( \Lambda = \sum_{J \subseteq \{1,2,3\}} \lambda_J \).

1. The 2-out-of-3: F system. Let \( \mathcal{O}_2 = \{12, 23, 13, 123\} \). Thus, \( \mathcal{E} - \mathcal{O}_2 = \{0, 1, 2, 3\} \) and

\[
A_{\{0,1,2,3\}} = \begin{bmatrix}
-\Lambda & \lambda_1 & \lambda_2 & \lambda_3 \\
0 & -\Lambda - \lambda_1 & 0 & 0 \\
0 & 0 & -\Lambda - \lambda_2 & 0 \\
0 & 0 & 0 & -\Lambda - \lambda_3
\end{bmatrix},
\]

and \( \alpha_{\mathcal{E} - \mathcal{O}_2} = (1, 0, 0, 0) \). Its reliability at time \( t \) is given by

\[
R_{\{2,3\}}(t) = (1, 0, 0, 0)e^{A_{\{0,1,2,3\}}t}e,
\]

and the MTTF is given by \(-{(1, 0, 0, 0)A_{\{0,1,2,3\}}^{-1}}e\).
2. The consecutive 2-out-of-3: F system. Let \( \mathcal{C}_2 = \{12, 23, 123\} \). Thus, \( \mathcal{E} - \mathcal{C}_2 = \{0, 1, 2, 3, 13\} \) and

\[
A_{\{0,1,2,3,13\}} = \begin{bmatrix}
-\Lambda & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_{13} \\
0 & -\Lambda - \lambda_1 & 0 & 0 & \lambda_3 + \lambda_{13} \\
0 & 0 & -\Lambda - \lambda_2 & 0 & 0 \\
0 & 0 & 0 & -\Lambda - \lambda_3 & \lambda_1 + \lambda_{13} \\
0 & 0 & 0 & 0 & -\Lambda - \sum_{J \subseteq \{1,3\}} \lambda_J
\end{bmatrix},
\]

and \( \alpha_{\mathcal{E} - \mathcal{C}_2} = (1, 0, 0, 0, 0) \). Its reliability at time \( t \) is given by

\[
R_{\mathcal{C}(2,3)}(t) = (1, 0, 0, 0, 0)e^{A_{\{0,1,2,3,13\}}t}e,
\]

and the MTTF is given by \(- (1, 0, 0, 0, 0)A_{\{0,1,2,3,13\}}^{-1}e \).

All these measures now can be easily calculated using Mathematica or Maple. For example, consider the following two cases.

1. Case 1: \( \lambda_1 = \lambda_2 = \lambda_3 = 4 \) and \( \lambda_L = 0 \) for any \( L \neq 1, 2, 3 \). This is the case that all three components are independent. In this case,

\[
A_{\{0,1,2,3\}} = \begin{bmatrix}
-12 & 4 & 4 & 4 \\
0 & -16 & 0 & 0 \\
0 & 0 & -16 & 0 \\
0 & 0 & 0 & -16
\end{bmatrix},
\]

and

\[
A_{\{0,1,2,3,13\}} = \begin{bmatrix}
-12 & 4 & 4 & 4 & 0 \\
0 & -16 & 0 & 0 & 4 \\
0 & 0 & -16 & 0 & 0 \\
0 & 0 & 0 & -16 & 4 \\
0 & 0 & 0 & 0 & -20
\end{bmatrix}.
\]

Thus,

\[
R_{(2,3)}(t) = (1, 0, 0, 0)e^{A_{\{0,1,2,3\}}t}e = 4e^{-12t} - 3e^{-16t}, t \geq 0,
\]

\[
R_{\mathcal{C}(2,3)}(t) = (1, 0, 0, 0, 0)e^{A_{\{0,1,2,3,13\}}t}e = 5e^{-12t} - 5e^{-16t} + e^{-20t}, t \geq 0.
\]
2. Case 2: $\lambda_L = 1$ for all $L \in \{1, 2, 3, 12, 13, 23, 123\}$ and $\lambda_0 = 0$. In this case, three components are positively dependent. In this case,

$$A_{\{0,1,2,3\}} = \begin{bmatrix}
-7 & 1 & 1 & 1 \\
0 & -8 & 0 & 0 \\
0 & 0 & -8 & 0 \\
0 & 0 & 0 & -8
\end{bmatrix},$$

and

$$A_{\{0,1,2,3,13\}} = \begin{bmatrix}
-7 & 1 & 1 & 1 & 1 \\
0 & -8 & 0 & 0 & 2 \\
0 & 0 & -8 & 0 & 0 \\
0 & 0 & 0 & -8 & 2 \\
0 & 0 & 0 & 0 & -10
\end{bmatrix}.$$

Thus,

$$R_{(2,3)}(t) = (1, 0, 0, 0)e^{A_{\{0,1,2,3\}}t}e = 4e^{-7t} - 3e^{-8t}, t \geq 0,$$

$$R_{C(2,3)}(t) = (1, 0, 0, 0, 0)e^{A_{\{0,1,2,3,13\}}t}e = \frac{17}{3}e^{-7t} - 5e^{-8t} + \frac{1}{3}e^{-10t}, t \geq 0.$$

The comparisons of $R_{(2,3)}(t)$ and $R_{C(2,3)}(t)$ in these two cases are illustrated in Figures 1 and 2.

Note that the marginal distribution of the $j$-th component of the Marshall-Olkin distributed random vector is exponential with mean $1/\sum_{S \subseteq \{3\}} \lambda_S$. Thus, in above two cases, the marginal distributions of two Marshall-Olkin distributions are the same, and only difference is that the distribution in Case 2 is more correlated than that in Case 1. It is easy to verify the following two facts.

1. The reliability of the series system of the components with the Marshall-Olkin distribution given in Case 2 is larger than that of the series system of components with the life distribution given in Case 1.

2. The reliability of the parallel system of the components with the Marshall-Olkin distribution given in Case 2 is smaller than that of the parallel system of components with the life distribution given in Case 1.

As illustrated in Figures 1 and 2, the reliability comparisons of the 2-out-of-3 systems exhibit more interesting patterns. Indeed, ignoring the failure dependence among the components often results in over-estimating or under-estimating the system reliability.
Figure 1: Comparison of 2-out-of-3 systems: The solid curve represents the reliability of a 2-out-of-3 system of independent components, and the dot curve represents the reliability for positively dependent components.
Figure 2: *Comparison of consecutive 2-out-of-3 systems*: The solid curve represents the reliability of a consecutive 2-out-of-3 system of independent components, and the dot curve represents the reliability for positively dependent components.
References


