WASHINGTON STATE UNIVERSITY
Department of Mathematics

# TECHNICAL REPORT 2004-8

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August 2004

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# On Two-Phase Stefan Problem Arising from a Microwave Heating Process

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**Abstract**: In this paper we study a free boundary problem modeling a phase-change process by using microwave heating technique. The mathematical model consists of Maxwell's equations coupled with nonlinear heat conduction with a phase-change. The enthalpy form is used to characterize the phase-change process in the model. It is shown that the problem has a global solution.

**Key Words and Phases**: Microwave heating, phase-change, global existence.

**AMS Subject Classifications**: 35R35.

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## 1. Introduction

Suppose that a material with solid and liquid phases occupies a bounded domain $\Omega \subset R^3$ with $C^1$-boundary $\partial\Omega$. If we heat up the material by using intense microwaves from the boundary $\partial\Omega$, the solid phase of the material will begin to melt.

To describe the above physical process, we introduce the electric and magnetic fields $\mathbf{E}(x,t)$ and $\mathbf{H}(x,t)$, respectively, in $\Omega$. Hereafter, a bold letter represents a vector or vector function in three-space dimensions. Then $\mathbf{E}(x,t)$ and $\mathbf{H}(x,t)$ satisfy the following well-known Maxwell equations (see [9]):

$$\varepsilon\mathbf{E}_t + \sigma\mathbf{E} = \nabla \times \mathbf{H}, \qquad (x,t) \in Q_T, \tag{1.1}$$

$$\mu\mathbf{H}_t + \nabla \times \mathbf{H} = 0, \qquad (x,t) \in Q_T, \tag{1.2}$$

where $Q_T = \Omega \times (0,T]$ and $\mathbf{J}(x,t) = \sigma\mathbf{E}$ is used by Ohm's law, $\varepsilon, \mu$ and $\sigma$ are the electric permittivity, magnetic permeability and the electric conductivity, respectively.

Let $u(x,t)$ be the temperature in $Q_T$. During the heating process, the local density of heat source generated by microwaves is equal to $\mathbf{E} \cdot \mathbf{J} = \sigma(x,u)|\mathbf{E}|^2$, where the electric conductivity $\sigma = \sigma(x,u)$ typically depends on $u$ such as (see [12, 13])

$$\sigma(x,u) = \frac{a(x)}{b(x) + c(x)u}, \text{or}\, \sigma(x,u) = a(x)e^{-b(x)u},$$

where $a(x), b(x)$ and $c(x)$ are positive functions. It also may have a jump discontinuity for the temperature changing from solid phase to liquid phase:

$$\sigma(x,u) = \begin{cases} \sigma_l(x,u), & \text{if } u(x,t) < m, \\ \sigma_s(x,u), & \text{if } u(x,t) > m, \end{cases}$$

where the subscript $l$ or $s$ represents the function in liquid or solid phase and $m$ is the melting temperature.

By using the enthalpy method, the temperature $u(x,t)$ satisfies the following heat equation in the weak sense (see [4, 5] and also Remark 1.1 below ):

$$A(u)_t - \nabla[k(x,u)\nabla u] = \sigma(x,u)|\mathbf{E}|^2, \qquad (x,t) \in Q_T, \tag{1.3}$$

where

$$A(u) = \begin{cases} u - 1, & \text{if } u < m, \\ [m-1, m], & \text{if } u = m, \\ u, & \text{if } u > m, \end{cases}$$

where the coefficient of heat conduction $k$ may be different in solid and liquid phases:

$$k(x,u) = \begin{cases} k_l(x,u), & \text{if } u(x,t) < m, \\ k_s(x,u), & \text{if } u(x,t) > m, \end{cases}$$

Because of the heat source in Eq.(1.3), the interface set $\Gamma_T = \{(x,t) \in Q_T : u(x,t) = m\}$ may have positive area, i.e. mushy region may exist. In this case one has to define the value of heat conductivity $k(x,u)$ and $\sigma(x,u)$ on $\Gamma_T$ and Eq.(1.2) is understood as an inclusion ([4]:

$$A(u)_t - \nabla[k(x,u)\nabla u] - \sigma(x,u)|\mathbf{E}|^2 \ni 0, \qquad (x,t) \in Q_T.$$

Define

$$\begin{aligned} &\sigma(x,m) \text{ is between the values } \sigma_l(x,m+) \text{ and } \sigma_s(x,m-) \text{ for any } x \in \Gamma_T, \\ &k(x,m) \text{ is between the values of } k_l(x,m+) \text{ and } k_s(x,m-) \text{ for any } x \in \Gamma_T, \end{aligned}$$

where $\sigma_l(x,m+) = \lim_{u\to m+} \sigma_l(x,u)$ and other quantities are defined similarly.

If $\sigma(x,u)$ and $k(x,u)$ are independent of $x$. Then one can simply define

$$\sigma(m) \in [\sigma_0, \sigma_1], k(m) \in [k_0, k_1],$$

where constants $k_0, k_1, \sigma_0$ and $\sigma_1$ are defined as follows:

$$\begin{aligned} k_0 &= min\{k_l(m), k_s(m)\}, k_1 = max\{k_l(m), k_s(m)\}, \\ \sigma_0 &= min\{\sigma_l(m), \sigma_s(m)\}, \sigma_1 = max\{\sigma_l(m), \sigma_s(m)\}. \end{aligned}$$

To complete the problem, we prescribe the following initial and boundary conditions:

$$\mathbf{N} \times \mathbf{E}(x,t) = \mathbf{N} \times \mathbf{G}(x,t), \qquad (x,t) \in S_T \tag{1.4}$$

$$u_n(x,t) = 0, \qquad (x,t) \in S_T, \tag{1.5}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.6}$$

where $\mathbf{G}(x,t)$ is given external vector function on $S_T = \partial\Omega \times [0,T]$, $\mathbf{N}$ is the outward normal on $S = \partial\Omega$, $u_n(x,t) := \nabla u \cdot \mathbf{N}$ is the normal derivative on $S$, $\mathbf{E}_0(x), \mathbf{H}_0(x)$ and $u_0(x)$ are the prescribed initial electric, magnetic fields and initial temperature.

The Stefan-type free boundary problems have been studied extensively by many researchers (see Monographs [6, 11, 18] and many conference proceedings). The classical enthalpy method is widely used to describe a phase-change process (see [1, 3, 4, 5, 11, 15, 16] etc. for examples). For microwave heating problems without phase-change, some research has been carried out (see [7, 8, 12, 13, 19] etc. and also see recent lecture notes [22] Chapter 6 for the theory). When a phase-change takes place, Coleman in [2] studied the microwave melting in one-space dimension and obtained some numerical solutions. In [17], Pangrie et al. uses time-harmonic Maxwell's equations and the enthalpy method to model the microwave melting process and obtained the numerical solution for a radially symmetric domain by using finite-difference method. In [20] the author studied a phase-change problem arising from microwave heating processes in one-space dimension, where a kinetic type of condition is given on the interface due to the superheating phenomenon. Global existence and uniqueness are established in [20]. We would also like to mention a related work on a phase-change problem for the induction heating ([21]) recently, where displacement current is neglected and magnetic field is assumed to be time-harmonic. However, none of the previous research deals with the fully coupled system (1.1)-(1.3) with phase-change. One of the difficulties is that there is not much known about the regularity of solutions to Maxwell's equations with variable coefficients. Another difficulty is that the nonlinear term $\sigma(x,u)|\mathbf{E}|^2$ which only belongs to $L^1(Q_T)$. Moreover, the electric conductivity may have a jump discontinuity from solid to liquid phase. In this paper we study the phase-change problem (1.1)-(1.6). By using similar ideas from [21] it is shown that under certain conditions on coefficients of the system (1.1)-(1.3) the problem (1.1)-(1.6) has a global weak solution. The global existence is also established for the case when the electric and magnetic fields are assumed to be time-harmonic. Moreover, for one-space dimension we prove the existence of a weak solution for $\sigma(x,t,u)$ with linear growth in $u$-variable. In this paper the uniqueness is left out as an open question even for one-space dimension.

This paper is organized as follows. In section 2, we prove that the problem (1.1)-(1.6) has a weak solution in $Q_T$ for any $T > 0$. In section 3, we study the problem for time-harmonic electric and magnetic fields and obtain the global solution for the problem. In section 4, we study the problem for one-space dimension and prove the existence of a weak solution for a more general function $\sigma(x,u)$.

## 2. Global Existence of Weak Solutions

In this section we first define weak solution to the problem and then consider an approximate problem by the standard approximation for $A(u)$ and $\sigma(u)$. It is shown that the approximated problem has a unique solution. Moreover, some uniform estimates for the approximate solution are derived. Finally, we establish the global existence to the problem (1.1)-(1.5) by using a compactness argument.

We list some basic assumptions for the coefficients and the known data.
H(2.1): (a)Let $\varepsilon(x)$ and $\mu(x)$ be in $L^\infty(\Omega)$ with a positive lower bound

$$0 < r_0 \leq \varepsilon(x), \mu(x) \leq R_0, \qquad x \in \Omega,$$

where $r_0$ and $R_0$ are positive constants.
(b) $\sigma(x,u)$ is non-negative and is bounded in $\Omega \times [M,\infty)$ for some large $M > 0$ and $\sigma_0$:

$$0 \leq \sigma(x,u) \leq \sigma_0, u\sigma(x,u) \leq \sigma_0, \qquad (x,u) \in \Omega \times [M,\infty).$$

(c) The functions $k_l(x,u)$ and $k_s(x,u)$ are of class $C^{1+\alpha}(\Omega \times R)$ and bounded with a positive lower bound:

$$0 < r_0 \leq k_s(x,u), k_l(x,u) \leq R_0, \qquad (x,u) \in \Omega \times [0,\infty).$$

H(2.2): (a) Let $u_0(x)$ be in $\in L^\infty(\Omega)$ and $\mathbf{E}_0(x), \mathbf{H}_0(x) \in L^2(\Omega)^3$.
(b) Let $\mathbf{G}(x,t) \in C([0,T]; H^{\frac{1}{2}}(S))$.

It is easy to see that the conditions on $\sigma(x,u)$ are satisfied for $\sigma(x,u) = \frac{1}{(1+u)^p}$ with $p \geq 1$ or $\sigma(x,u) = a(x)e^{-u}$. For reader's convenience, we recall some function spaces associated with Maxwell's equations. Other Sobolev spaces are the same as in [10]. Let

$$\begin{aligned} H(curl,\Omega) &= \{\mathbf{V} \in L^2(\Omega)^3 : \nabla \times \mathbf{V} \in L^2(\Omega)^3\}; \\ H_0(curl,\Omega) &= \{\mathbf{V} \in H(curl,\Omega) : \mathbf{N} \times \mathbf{V} = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

$H(curl,\Omega)$ is a Hilbert space equipped with inner product

$$(\mathbf{V},\mathbf{K}) = \int_\Omega [\mathbf{V} \cdot \mathbf{K} + (\nabla \times \mathbf{V}) \cdot (\nabla \times \mathbf{K})]dx.$$

**Definition 2.1**: A triple of functions $(\mathbf{E}(x,t), \mathbf{H}(x,t), u(x,t))$ is said to be a weak solution to the problem (1.1)-(1.6), if

$$\mathbf{E}(x,t), \mathbf{H}(x,t) \in C([0,T]; L^2(\Omega)),$$

and $u(x,t) \in L^2(0,T; H^1(\Omega)) \bigcap C([0,T]; L^2(\Omega))$, which satisfy the following integral identities:

$$\begin{aligned}
&\int_0^T \int_\Omega [-\varepsilon \mathbf{E} \cdot \mathbf{\Psi}_t + \sigma \mathbf{E} \cdot \mathbf{\Psi}] dxdt = \int_0^T \int_\Omega [\mathbf{H} \cdot (\nabla \times \mathbf{\Psi})] dxdt + \int_\Omega \varepsilon \mathbf{E}_0 \cdot \mathbf{\Psi}(x,0) dx, \\
&\int_0^T \int_\Omega [-\mu \mathbf{H} \cdot \mathbf{\Phi}_t + \mathbf{E} \cdot (\nabla \times \mathbf{\Phi})] dxdt = \int_\Omega [\mu \mathbf{H}_0(x) \cdot \mathbf{\Phi}(x,0)] dx, \\
&\int_0^T \int_\Omega [-A(u)\psi_t + k(x,u) \nabla u \nabla \psi] dxdt = \int_0^T \int_\Omega \sigma(x,u) |\mathbf{E}|^2 \psi dxdt + \int_\Omega A(u_0) \psi dx
\end{aligned}$$

for any test vector functions $\mathbf{\Psi}, \mathbf{\Phi} \in L^2(0,T; H_0(curl, \Omega)) \bigcap C([0,T]; L^2(\Omega)^3)$ and any test function $\psi \in H^1(0,T; H^1(\Omega))$ with $\mathbf{\Psi}(x,T) = \mathbf{\Phi}(x,T) = 0$ and $\psi(x,T) = 0$ on $\Omega$.

Since the weak solution $\mathbf{E}(x,t) \in L^2(\Omega)$, we have to specify the boundary condition (1.4) in the weak sense. Note that

$$\mathbf{H}(x,t) = \mathbf{H}_0(x) - \frac{1}{\mu(x)} \int_0^t \nabla \times \mathbf{E}(x,\tau) d\tau = \mathbf{H}_0(x) - \frac{1}{\mu(x)} \nabla \times \mathbf{W}(x,t),$$

where

$$\mathbf{W}(x,t) = \int_0^t \mathbf{E}(x,\tau) d\tau.$$

It follows that $\mathbf{H} \in L^2(\Omega)^3$ implied $\nabla \times \mathbf{W} \in L^2(\Omega)^3$ for each a.e. fixed $t \in [0,T]$. Consequently, the trace $\mathbf{N} \times \mathbf{W}(x,t)$ is well-defined on $\partial\Omega$. We define

$$\mathbf{N} \times (\mathbf{E}(x,t) - \mathbf{G}(x,t)) = 0, \qquad (x,t) \in S_T$$

if and only if

$$\mathbf{N} \times [\mathbf{W}(x,t) - \int_0^t \mathbf{G}(x,\tau) d\tau] = 0, \qquad (x,t) \in S_T.$$

Introduce a new function,

$$U(x,t) := K(x,u) = \int_m^u k(x,s) ds, \qquad (x,t) \in Q_T.$$

Then the assumption H(2.1)(b) implies that the inverse function $u(x,t) = K^{-1}(x,U)$ exists. Consequently, Eq. (1.3) can be written in the weak sense as follows:

$$A^*(x,U)_t - \Delta U = \sigma^*(x,U)|\mathbf{E}|^2, \qquad (x,t) \in Q_T,$$

where

$$A^*(x,U) = \begin{cases} K_s^{-1}(x,U) - 1, & \text{if } U < 0, \\ [-1,0], & \text{if } U = 0, \\ K_l^{-1}(x,U), & \text{if } U > 0, \end{cases}$$

and $K_s^{-1}(x,U), K_l^{-1}(x,U)$ are the inverse functions of $K_s(x,u), K_l(x,u)$, respectively. Moreover,

$$\sigma^*(x,U) = \begin{cases} \sigma(x, K_s^{-1}(x,U)), & \text{if } U < 0, \\ [\sigma_0, \sigma_1], & \text{if } U = 0, \\ \sigma(x, K_l^{-1}(x,U)), & \text{if } U > 0. \end{cases}$$

¿From now on, instead of using $U(x,t), A^*(x,U)$ and $\sigma^*(x,U)$ we will continue to use notation $u(x,t), A(x,u)$ and $\sigma(x,u)$ for simplicity. By assumption H(2.2), there exists an extension for $\mathbf{G}(x,t)$ such that $\mathbf{G}(x,t) \in C([0,T]; H^1(\Omega)^3)$. Moreover, from the assumption of H(2.1)(c) there exists a constant $a_0 > 0$ such that $A'(x,u) := A_u(x,u) \geq a_0$ for all $(x,u) \in \Omega \times R$ whenever $u \neq m$.

Let $A_n(x,u)$ and $\sigma_n(x,u)$ be the smooth approximation of $A(x,u)$ and $\sigma(x,u)$, respectively. Moreover, we require that

$$\begin{aligned} &A_n(x,u) = A(x,u), \sigma_n(x,u) = \sigma(x,u), \text{if } |u-m| \geq \tfrac{1}{n}, \\ &A_n'(x,u) \geq \frac{r_0}{2}, A_n(x,u) \to A(x,u), \sigma_n(x,u) \to \sigma(x,u) \end{aligned}$$

strongly in $L^2(\Omega \times [-M,M])$ for some large $M > 0$ as $n \to \infty$ . We also make a smooth approximation for $u_0(x)$, denoted by $u_{0n}(x)$, such that $\nabla u_{0n}(x) = 0$ on $S$ and $u_{0n}(x) \to u_0(x)$ strongly in $L^2(Q_T)$.

Consider the following approximate system:

$$\varepsilon(x)\mathbf{E}_t + \sigma_n(x,u)\mathbf{E} = \nabla \times \mathbf{H}, \qquad (x,t) \in Q_T, \tag{2.1}$$

$$\mu(x)\mathbf{H}_t + \nabla \times \mathbf{H} = 0, \qquad (x,t) \in Q_T, \tag{2.2}$$

$$A_n(x,u)_t - \Delta u = \sigma_n(x,u)|\mathbf{E}|^2, \qquad (x,t) \in Q_T, \tag{2.3}$$

$$\mathbf{N} \times \mathbf{E}(x,t) = \mathbf{N} \times \mathbf{G}(x,t), \qquad (x,t) \in S_T, \tag{2.4}$$

$$u_n(x,t) = 0, \qquad (x,t) \in S_T, \tag{2.5}$$

$$\mathbf{E}(x,0) = \mathbf{E}_0(x), \mathbf{H}(x,0) = \mathbf{H}_0(x), u(x,0) = u_{0n}(x), x \in \Omega. \tag{2.6}$$

¿From Theorem 2.1 ([19]), the problem (2.1)-(2.6) has a unique weak solution

$$(\mathbf{E}_n(x,t), \mathbf{H}_n(x,t)) \in C([0,T]; H_0(curl,\Omega)) \times C([0,T]; H(curl,\Omega))$$

and

$$u_n(x,t) \in C([0,T]; L^2(\Omega)) \bigcap L^2(0,T; H^1(\Omega)).$$

Moreover,

$$\mathbf{E}_n(x,t) - \mathbf{G}(x,t) \in L^\infty(0,T; H_0(curl,\Omega)).$$

Furthermore, since $u_0(x) \in L^\infty(\Omega)$ and $\sigma_n(x,u)|\mathbf{E}|^2 \geq 0$, it follows from the maximum principle that there exists a constant $M_0 > 0$ independent of $n$ such that $u_n(x,t) \geq -M_0$ on $Q_T$.

Now we derive some uniform estimates.

**Lemma 2.1**: There exist constant $C_1$ such that

$$\sup_{0 \leq t \leq T} \int_\Omega [|\mathbf{E}_n|^2 + |\mathbf{H}_n|^2] dx \leq C_1,$$

where $C_1$ depends only on the known data.

**Proof**: For simplicity, we shall drop the subscript $n$ for the solution $(\mathbf{E}_n, \mathbf{H}_n, u_n)$ whenever without causing confusion. To derive the estimate, we take the inner product by $\mathbf{E}(x,t) - \mathbf{G}$ to Eq. (2.1) and by $\mathbf{H}(x,t)$ to Eq. (2.2), respectively, to obtain:

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_\Omega [\varepsilon|\mathbf{E}|^2] dx + \int_\Omega \sigma(x,u)|\mathbf{E}|^2 dx \\
&= \int_\Omega \varepsilon \mathbf{E} \cdot \mathbf{G} dx + \int_\Omega [\sigma \mathbf{E} \cdot \mathbf{G}] dx + \int_\Omega [\nabla \times \mathbf{H} \cdot (\mathbf{E} - \mathbf{G})] dx, \\
&\frac{d}{dt}\frac{1}{2}\int_\Omega [\mu|\mathbf{H}|^2] dx + \int_\Omega [\nabla \times \mathbf{E} \cdot \mathbf{H}] dx = 0.
\end{aligned}$$

We add up the above equations and use the fact,

$$\int_\Omega \nabla \times \mathbf{H} \cdot (\mathbf{E} - \mathbf{G}) dx = \int_\Omega \mathbf{H} \cdot [\nabla \times (\mathbf{E} - \mathbf{G})] dx,$$

to obtain

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_\Omega[\varepsilon|\mathbf{E}|^2+\mu|\mathbf{H}|^2]dx+\int_\Omega\sigma(x,u)|\mathbf{E}|^2dx\\&\le C\int_\Omega[|\mathbf{E}|^2+|\mathbf{H}|^2]dx+C\int_\Omega[|\mathbf{G}|^2+|\nabla\times\mathbf{G}|^2]dx,\end{aligned}$$

where $C$ depends only on $L^\infty$-bounds of $\varepsilon(x), \mu(x)$ and $\sigma(x,u)$, but independent of $n$.

Gronwall's inequality yields the desired estimate.

Q.E.D.

**Lemma 2.2**: There exists a constant $C_2$ such that

$$\sup_{0\le t\le T}\int_\Omega|u_n|^2dx+\int\int_{Q_T}|\nabla u_n|^2dxdt\le C_2,$$

where $C_2$ depends only on known data.

**Proof**: Since $A_n'(x,u)\ge\frac{r_0}{2}$, the inverse of the function for $v(x,t):=A_n(x,u)$ exists, denoted by $u=B_n(x,v)$. Then

$$\begin{aligned}&\int_0^t\int_\Omega A_n(u)_t u dxdt=\int_0^t[\frac{d}{dt}\int_\Omega\int_0^v B_n(x,s)dsdx]dt\\&=\int_\Omega\int_0^v B_n(x,s)dsdx-\int_\Omega\int_0^{A_n(u_0)}B_n(x,s)dsdx\\&\ge b_0\int_\Omega u^2dx-C,\end{aligned}$$

for some $b_0>0$ since $k_0s\le B_n(x,s)\le k_1(s+1)$ from the assumption H(2.1)(c).

On the other hand, it is clear that

$$-\int_0^t\int_\Omega(\Delta u)udxdt=\int_0^t\int_\Omega|\nabla u|^2dxdt.$$

Moreover, by Lemma 2.1 and the assumption H(2.1) we have

$$\int_\Omega\sigma(x,u)u|\mathbf{E}|^2dx\le C,$$

where $C$ depends only on known data.

We sum up the above estimates to obtain

$$\int_\Omega u^2dx+\int_0^t\int_\Omega|\nabla u|^2dxdt\le C_2,$$

where $C_2$ depends only on known data.

Q. E. D.

To prove the existence of a weak solution for the problem (1.1)-(1.6), we need the following lemma from [19].

**Lemma 2.3**: Suppose $\sigma_n(x,t) \to \sigma(x,t)$ strongly in $L^2(Q_T)$. Let $(\mathbf{E}_n(x,t), \mathbf{H}_n(x,t))$ be the solution of the Maxwell equations:

$$\begin{aligned}
&\varepsilon \mathbf{E}_t + \sigma_n(x,t)\mathbf{E} = \nabla \times \mathbf{H}, && (x,t) \in Q_T,\\
&\mu \mathbf{H}_t + \nabla \times \mathbf{E} = 0, && (x,t) \in Q_T,\\
&\mathbf{N} \times \mathbf{E} = \mathbf{N} \times \mathbf{G}, && (x,t) \in \partial\Omega \times (0,T],\\
&\mathbf{E}(x,0) = \mathbf{E}_0(x), \mathbf{H}(x,0) = \mathbf{H}_0(x), && x \in \Omega.
\end{aligned}$$

Let $(\mathbf{E}(x,t), \mathbf{H}(x,t))$ be the solution of the above Maxwell equations where $\sigma_n(x,t)$ is replaced by $\sigma(x,t)$. Then $(\mathbf{E}_n(x,t), \mathbf{H}_n(x,t))$ converges to $(\mathbf{E}(x,t), \mathbf{H}(x,t))$ strongly in $L^2(Q_T)$.

Q.E.D.

**Theorem 2.4**: The problem (1.1)-(1.6) possesses at least one weak solution in $Q_T$ for any $T > 0$.

**Proof**: From Lemma 2.1-2.2 and the weak compactness, we know, after extracting a subsequence if necessary, that

$$\begin{aligned}
&\mathbf{E}_n(x,t) \to \mathbf{E}(x,t), \mathbf{H}_n(x,t) \to \mathbf{H}(x,t) \text{ in weak-* } L^\infty(0,T;L^2(\Omega)^3),\\
&u_n(x,t) \to u(x,t) \text{ weakly in } L^2(0,T;H^1(\Omega)).
\end{aligned}$$

Moreover, by applying the result of Lemma 5.1 from [15] we see that $u_n(x,t)$ converges to $u(x,t)$ strongly in $L^2(Q_T)$ and almost everywhere in $Q_T$.

We multiply Eq.(2.1) and Eq.(2.2) by any test vector functions $\mathbf{\Psi}(x,t)$ and $\mathbf{\Phi}(x,t)$, respectively, in $H^1(0,T;H_0(curl,\Omega))$ to obtain

$$\begin{aligned}
&\int_0^T \int_\Omega [-\varepsilon \mathbf{E}_n \cdot \mathbf{\Psi}_t + \sigma_n(x,u_n)\mathbf{E}_n \cdot \mathbf{\Psi}] dxdt\\
&= \int_0^T \int_\Omega [\mathbf{H}_n \cdot (\nabla \times \mathbf{\Psi}) dxdt + \int_\Omega \varepsilon \mathbf{E}_0(x) \cdot \mathbf{\Psi}(x,0) dx,\\
&\int_\Omega [-\mu \mathbf{H}_n \cdot \mathbf{\Phi}_t + \mathbf{E}_n \cdot (\nabla \times \mathbf{\Phi})] dxdt = \int_\Omega [\mu \mathbf{H}_0(x) \cdot \mathbf{\Phi}(x,0)] dx.
\end{aligned}$$

After taking limit as $n \to \infty$, we see that $(\mathbf{E}, \mathbf{H})$ satisfies the integral identities in Definition 2.1 if we can prove $\mathbf{J}_n(x,t) := \sigma_n(x,u_n)\mathbf{E}_n$ converges to $\mathbf{J}(x,t) = \sigma(x,u)\mathbf{E}(x,t)$ weakly in $L^2(Q_T)$. We skip the proof here since it can be done by using the same technique as for a more complicated term $\sigma_n(x,u_n)|\mathbf{E}_n|^2$ below (see detailed proof below). Moreover, since $\mathbf{E}_n(x,t) - \mathbf{G}(x,t) \in L^\infty(0,T; H_0(curl,\Omega))$ and $\mathbf{H}_n(x,t) \in L^\infty([0,T]; L^2(\Omega))$, it follows that $\nabla \times \mathbf{W}_n(x,t) \in L^\infty([0,T], L^2(\Omega))$, where

$$\mathbf{W}_n(x,t) = \int_0^t \mathbf{E}_n(x,\tau)d\tau.$$

Thus, the trace $\mathbf{N} \times \mathbf{W}_n$ is well defined on $\partial\Omega$. Since $\mathbf{N} \times \mathbf{W}_n = \mathbf{N} \times \int_0^t \mathbf{G}(x,\tau)d\tau$ and $\mathbf{N} \times [\mathbf{E}_n - \mathbf{G}] = 0$ is equivalent to $\mathbf{N} \times [\mathbf{W}_n - \int_0^t \mathbf{G}(x,\tau)d\tau] = 0$. It follows that the boundary condition (1.4) holds.

For any small $\gamma > 0$ by Egorof's theorem there exists a subset $Q \subset Q_T$ with $|Q_T \backslash Q| < \gamma$ such that $u_n(x,t)$ converges to $u(x,t)$ uniformly on $Q$. Set $Q_\gamma = \{(x,t) \in Q : |u(x,t) - m| > \gamma\}$. Then, for $(x,t) \in Q_\gamma$, if $n$ is sufficiently large,

$$|u_n(x,t) - m| \geq \frac{\gamma}{2} \geq \frac{1}{n}.$$

On the other hand, for any $(x,t) \in Q \backslash Q_\gamma$

$$|u_n - u| \leq |u_n - m| + |u - m| \leq 2\gamma,$$

provided that $n$ is large enough.

Let $\phi$ be a smooth test vector function.

$$\begin{aligned}
&\int\int_{Q_T} [\sigma_n(x,u_n)|\mathbf{E}_n|^2 - \sigma(x,u)|\mathbf{E}|^2]\phi dxdt \\
&= \int\int_Q [\sigma_n(x,u_n)|\mathbf{E}_n|^2 - \sigma(x,u)|\mathbf{E}|^2]\phi dxdt + \\
&\int\int_{Q_T \backslash Q} [\sigma_n(x,u_n)|\mathbf{E}_n|^2 - \sigma(x,u)|\mathbf{E}|^2]\phi dxdt \\
&:= I_1 + I_2.
\end{aligned}$$

It is clear that $I_2 \to 0$ as $\gamma \to 0$ since $|Q_T \backslash Q| < \gamma$.

$$\begin{aligned}
I_1 \quad := \quad & \int\int_{Q_\gamma} [\sigma(x,u_n)|\mathbf{E}_n|^2 - \sigma(x,u)|\mathbf{E}|^2]\phi dxdt \\
& + \int\int_{Q \backslash Q_\gamma} [\sigma_n(x,u_n)|\mathbf{E}_n|^2 - \sigma(x,u)|\mathbf{E}|^2]\phi dxdt := J_1 + J_2.
\end{aligned}$$

$$\begin{aligned}|J_1| &\leq |\int\int_{Q_\gamma} \sigma(x,u_n)[|\mathbf{E}_n|^2 - |\mathbf{E}|^2]|\phi|dxdt| + \int\int_{Q_\gamma} |\sigma(x,u_n) - \sigma(x,u)|\mathbf{E}|^2]|\phi|dxdt \\ &:= J_{11} + J_{12}.\end{aligned}$$

It is clear that $J_{11} \to 0$ since $\mathbf{E}_n$ converges to $\mathbf{E}$ strongly in $L^2(Q_T)$ by Lemma 2.3 and $\sigma(x,u_n)$ is bounded. On the other hand,

$$Q_\gamma = [Q_\gamma \bigcap \{(x,t) \in Q : u(x,t) \geq m+\gamma\}] \bigcup [Q_\gamma \bigcap \{(x,t) \in Q : u(x,t) \leq m-\gamma\}].$$

Since $u_n \to u(x,t)$ a.e. on $Q_T$ and uniformly in $Q$, it follows that on $Q_\gamma \bigcap \{(x,t) : u(x,t) \geq m+\gamma\}$, $\sigma(x,u_n) = \sigma_l(x,u_n) \to \sigma_l(x,u)$ a.e. and on $Q_\gamma \bigcap \{(x,t) : u(x,t) \leq m-\gamma\}$, $\sigma(x,u_n) = \sigma_s(x,u_n) \to \sigma_s(x,u)$ a.e. as $n \to \infty$, by dominated convergence theorem we see $J_{12} \to 0$ as $n \to \infty$.

Next we prove $J_2 \to 0$ as $n \to \infty$. Without loss of generality, we assume

$$\sigma_s(x,m) \leq \sigma_l(x,m), \qquad x \in \Omega.$$

Then the approximation sequence $\sigma_n(x,u)$ can be assumed to be increasing in $u$-vaiable in a neighborhood of $u = m$. On $Q \backslash Q_\gamma$,

$$m - \gamma \leq u_n(x,t) \leq m + \gamma,$$

As $\sigma_n(x,u)$ is increasing in a neighborhood of $u = m$, we see

$$\sigma_n(x, m-\gamma) \leq \sigma_n(x,u_n) \leq \sigma_n(x, m+\gamma).$$

If $n$ is sufficiently large,

$$\sigma_n(x,m-\gamma) = \sigma_s(x,m-\gamma), \sigma_n(x,m+\gamma) = \sigma_l(m+\gamma).$$

By the weak convergence and the definition of $\sigma(x,m)$, when $\gamma \to 0, n \to \infty$ we have

$$\sigma_s(x,m-) \leq \sigma(x,m) \leq \sigma(x,m+), \qquad a.e. x \in \Omega.$$

It follows that

$$\begin{aligned}|J_2| &\leq |\int\int_{Q\backslash Q_\gamma} [\sigma_n(x,u_n)(|\mathbf{E}_n|^2 - |\mathbf{E}|^2)|\phi|dxdt| \\ &\quad +|\int\int_{Q\backslash Q_\gamma} (\sigma_n(x,u_n) - \sigma(x,u)|\mathbf{E}|^2]\phi)dxdt| \to 0\end{aligned}$$

as $n \to \infty, \gamma \to 0$.

Next we consider the convergence for $A_n(x, u_n)$. The weak compactness implies that there exists a function $\beta(x,t) \in L^2(Q_T)$ such that $A_n(u_n) \to \beta(x,t)$ weakly in $L^2(Q_T)$. We need to prove that the graph of $\beta(x,t) \in A(u)$ a.e. on $Q_T$. From the construction of $A_n$ and convergence of $u_n$ we see

$$\beta(x,t) = A(u) \text{ a.e. on } Q_\gamma.$$

On $Q_T \backslash Q_\gamma$,

$$m - \frac{2}{n} \leq u_n \leq m + \frac{2}{n}.$$

The monotonicity of $A(u)$ implies

$$A_n(m - \frac{2}{n}) \leq A_n(u_n) \leq A_n(m + \frac{2}{n}),$$

which is the same as

$$A(m - \frac{2}{n}) \leq A_n(u_n) \leq A(m + \frac{2}{n}).$$

The weak convergence yields that for a.e. $(x,t) \in Q_T \backslash Q_\gamma$

$$m - 1 \leq \beta(x,t) \leq m.$$

Now we multiply Eq. (2.3) by any test function $\psi \in H^1(0,T; H^1(\Omega))$ with $\psi(x,T) = 0$ and then integrate over $Q_T$ to obtain

$$\begin{aligned}&\int_0^T \int_\Omega [-A_n(x,u_n)\psi_t + \nabla u_n \nabla \psi] dx \\ &= \int_0^T \int_\Omega \sigma(x,u_n)|\mathbf{E}_n|^2 \psi dx dt + \int_\Omega A(x, u_{0n}(x))\psi(x,0) dx.\end{aligned}$$

Finally, we take limit as $n \to \infty$ to see that $(\mathbf{E}(x,t), \mathbf{H}(x,t), u(x,t))$ is indeed a weak solution of the problem (1.1)-(1.6). Q.E.D.

## 3. Global Existence in Time-Harmonic Fields

For some industrial applications (see [12, 13]), the time scale for electromagnetic field and the heat conduction is quite different. It is often to assume that the electric and magnetic fields are time-harmonic. This leads to the following problem:

$$i\mu\omega \mathbf{H} + \nabla \times \mathbf{E} = 0, \qquad x \in \Omega, \tag{3.1}$$

$$(i\varepsilon\omega + \sigma)\mathbf{E} = \nabla \times \mathbf{H}, \qquad x \in \Omega, \tag{3.2}$$

where $i$ represents the complex unit and $\omega$ is the frequency.

For many applied problems, it is often convenient to use a unified approach by assuming that (see [12]):

$$\varepsilon(x) = \varepsilon_1(x) + i\varepsilon_2(x), \mu(x) = \mu_1(x) - i\mu_2(x),$$

where $\varepsilon_1(x), \varepsilon_2(x), \mu_1(x)$ and $\mu_2(x)$ are positive functions.

It is clear that the system (3.1)-(3.2) is equivalent to the following one:

$$\nabla \times [\gamma(x)\nabla \times \mathbf{E}] + r(x,u)\mathbf{E} = 0, \qquad x \in \Omega, \tag{3.3}$$

where

$$\begin{aligned} \gamma(x) &:= \frac{1}{\mu(x)} = \frac{\mu_1}{\sqrt{|\mu_1|^2 + |\mu_2|^2}} + i\frac{\mu_2}{\sqrt{|\mu_1|^2 + |\mu_2|^2}}, \\ r(x,u) &:= i\omega(i\varepsilon(x)\omega + \sigma(x,u)). \end{aligned}$$

Consider the phase-change problem:

$$\nabla \times [\gamma(x)\nabla \times \mathbf{E}] + r(x,u)\mathbf{E} = 0, \qquad x \in \Omega, \tag{3.4}$$

$$A(x,u)_t - \Delta u = \sigma(x,u)|\mathbf{E}|^2, \qquad (x,t) \in Q_T, \tag{3.5}$$

$$\mathbf{N} \times \mathbf{E}(x) = \mathbf{N} \times \mathbf{G}(x), \qquad x \in \partial\Omega, \tag{3.6}$$

$$u_n(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T], \tag{3.7}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega. \tag{3.8}$$

Note that the coefficient $\sigma(x,u)$ depends on $t$ since $u(x,t)$ is a function of $t$. The solution $\mathbf{E}$ is also a function of $t$. However, this time variable for the heat conduction is different from the time-variable in electromagnetic field.

A weak solution to (3.4)-(3.8) can be defined as follows.

**Definition 3.1**: A pair functions $(\mathbf{E}(x,t), u(x,t))$ is called a weak solution of (3.4)-(3.8) if $\mathbf{E}(x,t) \in H(curl,\Omega), u(x,t) \in L^2(0,T;H^1(\Omega))$ with $\mathbf{N} \times (\mathbf{E} - \mathbf{G}) \in H_0(curl,\Omega)$ and the following integral identities hold:

$$\int_\Omega [\gamma(\nabla \times \mathbf{E}) \cdot (\nabla \times \mathbf{\Psi}) + r(x,u)\mathbf{E} \cdot \mathbf{\Psi}]dx = 0,$$

$$\int\int_{Q_T} [-A(x,u)\psi_t + \nabla u \nabla \psi]dxdt = \int\int_{\Omega_T} \sigma(x,u)|\mathbf{E}|^2\psi dxdt + \int_\Omega A(x,u_0)\psi(x,0)dx$$

for any test vector functions $\mathbf{\Psi}, \in H_0(curl, \Omega)$ and any test function $\psi \in H^1(0,T; H^1(\Omega))$ with $\psi(x,T) = 0$ on $\Omega$.

H(3.1): (a) Let $\varepsilon_1(x), \varepsilon_2(x), \mu_1(x)$ and $\mu_2(x)$ be nonnegative and of class $L^\infty(\Omega)$ with $\varepsilon_1 \geq r_0, \varepsilon_2 \geq 0$. Let

$$0 \leq \sigma(x,u) \leq \sigma_0,\ u\sigma(x,u) \leq \sigma_1, u \in [M, \infty).$$

Moreover, there exists a constant $\sigma_1$ such that

$$\sigma(x,u) - |\varepsilon_2|_{L^\infty(\Omega)} \geq \sigma_1 > 0.$$

(b) Let $\mathbf{G}(x) \in H(curl, \Omega)$.
H(3.2): Let $A(x,u)$ be defined as in section 2 and satisfy the same condition as in H(2.1)(c). Moreover, $u_0(x) \in L^2(\Omega)$ and nonnegative.

**Theorem 3.1** Under the assumptions H(3.1)-(3.3) the problem (3.4)-(3.8) has a global weak solution.
**Proof**: As the proof is quite similar to that of Theorem 2.4, we only give an outline.
Step 1: Approximating the problem. By constructing smooth approximation for $\sigma$ and $A(x,u)$, we consider the approximation problem:

$$\nabla \times [\gamma(x)\nabla \times \mathbf{E}] + r_n(x,u)\mathbf{E} = 0, \qquad x \in \Omega, \tag{3.9}$$

$$A_n(x,u)_t - \Delta u = \sigma_n(x,u)|\mathbf{E}|^2, \qquad (x,t) \in Q_T, \tag{3.10}$$

$$\mathbf{N} \times \mathbf{E} = \mathbf{N} \times \mathbf{G}, \qquad x \in \partial\Omega, \tag{3.11}$$

$$u_n(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T], \tag{3.12}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega. \tag{3.13}$$

This problem has at least one weak solution $(\mathbf{E}_n(x,t), u_n(x,t))$ ([21]).
Step 2: Deriving uniform estimates.
There exist constants $C_1$ and $C_2$ independent of $n$ such that

$$\int_\Omega |\nabla \times \mathbf{E}_n|^2 dx + \int_\Omega |\mathbf{E}_n|^2 dx \leq C_1,$$

$$\sup_{0\leq t\leq T} \int_\Omega u_n^2 dx + \int\int_{Q_T} |\nabla u_n|^2 dxdt \leq C_2.$$

To prove the first estimate, we take the inner product by $(\mathbf{E}-\mathbf{G})^*$, the complex conjugate of $\mathbf{E}-\mathbf{G}$, to Eq.(3.9) to obtain

$$\int_\Omega \gamma(\nabla\times\mathbf{E})\cdot[\nabla\times(\mathbf{E}-\mathbf{G})^*]+r_n(x,u)\mathbf{E}\cdot(\mathbf{E}-\mathbf{G})^*]dx=0. \tag{3.14}$$

We first take the imaginary part of the above equation to obtain

$$\int_\Omega \frac{\mu_2}{\sqrt{\mu_1^2+\mu_2^2}}|\nabla\times\mathbf{E}|^2dx+\int_\Omega(\sigma-\varepsilon_2)|\mathbf{E}|^2dx\le C,$$

where the constant $C$ depends only on known data. By H(3.1), we obtain

$$\int_\Omega |\mathbf{E}|^2dx\le\frac{C}{\sigma_1}.$$

Now we take the real part of Eq. (3.15) and use the assumption H(3.1) again to obtain

$$\int_\Omega |\nabla\times\mathbf{E}|^2dx\le C,$$

where $C$ depends only on known data.

The second estimate is the same as Lemma 2.3.

Step 3: Taking the limit. This step is almost identical to that of Theorem 2.4, we skip it here.

Q.E.D.

**Remark 3.1**: The assumption H(3.1) is only one type of sufficient conditions to ensure the global existence of a unique weak solution to time-harmonic Maxwell's equations. Other types of sufficient conditions can be found in [14, 22].

## 4. One-dimensional Problem

In this section we study the problem (1.1)-(1.5) in one space dimension and prove that the weak solution exists globally for $\sigma(x,t,u)$ with linear growth.

Let $Q_T=\{(x,t):0<x<1,0<t<T\}$. For one-space dimension, we assume $\mathbf{E}(x,t)=\{0,e(x,t),0\},\mathbf{H}(x,t)=\{0,0,h(x,t)\}$. Then the system (1.1)-(1.3) becomes the following form:

$$\varepsilon(x)e_t+\sigma(x,t,u)e=-h_x,\qquad (x,t)\in Q_T, \tag{4.1}$$

$$\mu(x)h_t+e_x=0,\qquad (x,t)\in Q_T, \tag{4.2}$$

$$A(x,u)_t-u_{xx}=\sigma(x,t,u)|e(x,t)|^2,\qquad (x,t)\in Q_T, \tag{4.3}$$

where $A(x,u)$ is the same as in Section 2.

By solving $h(x,t)$ from Eq.(4.2), we see

$$h(x,t) = h_0(x) - \frac{1}{\mu(x)} w_x(x,t), \qquad (x,t) \in Q_T,$$

where

$$w(x,t) = \int_0^t e(x,\tau) d\tau.$$

For simplicity, we assume $h_0(x) = 0$ on $Q_T$. It follows that Eq.(4.1)-(4.3) is equivalent to the following system:

$$\varepsilon(x) w_{tt} - (\gamma(x) w_x)_x + \sigma(x,t,u) w_t = 0, \tag{4.4}$$

$$A(x,u)_t - u_{xx} = \sigma(x,t,u)|w_t(x,t)|^2, \qquad (x,t) \in Q_T. \tag{4.5}$$

The initial and boundary conditions are as follows:

$$w(0,t) = f_1(t), w(1,t) = f_2(t), u_x(0,t) = u_x(1,t) = 0, t \in [0,T], \tag{4.6}$$

$$w(x,0) = 0, w_t(x,0) = e_0(x), u(x,0) = u_0(x), \qquad 0 < x < 1. \tag{4.7}$$

H(4.1): (a) Let $\varepsilon(x), \gamma(x)$ satisfies H(2.1)(a). Let $\sigma(x,t,u)$ satisfies

$$0 \le \sigma(x,t,u) \le \sigma_0(1+u), (x,t,u) \in Q_T \times [0,\infty).$$

(b) Let $A(x,u)$ be defined as in section 2 and satisfy the same condition as in H(2.1)(c).
(c) Let $f_1(t), f_2(t) \in H^1(0,T)$ with $f_1(0) = f_2(0) = 0$ and $e_0(x), u_0(x) \in L^2(0,1)$.

**Theorem 4.1**: Under the assumption H(4.1) the problem (4.4)-(4.7) has a weak solution globally.

**Proof**: As for $n-$dimensional case, we make a smooth approximation for $\sigma(x,t,u)$ and $A(x,u)$ and then consider the following approximate problem:

$$\varepsilon(x) w_{tt} - (\gamma(x) w_x)_x + \sigma_n(x,t,u) w_t = 0, \qquad (x,t) \in Q_T, \tag{4.8}$$

$$A_n(x,u)_t - u_{xx} = \sigma_n(x,t,u)|w_t(x,t)|^2, \qquad (x,t) \in Q_T, \tag{4.9}$$

$$w(0,t) = f_1(t), w(1,t) = f_2(t), u_x(0,t) = u_x(1,t) = 0, t \in [0,T], \tag{4.10}$$

$$w(x,0) = 0, w_t(x,0) = e_0(x), u(x,0) = u_0(x), \qquad 0 < x < 1. \tag{4.11}$$

This problem has a unique weak solution ([19])

$$(w_n(x,t), u_n(x,t)) \in H^1(0,T;H^1(0,1)) \times L^2(0,T;H^1(0,1)).$$

Again we will omit the subscript $n$. Now we derive some uniform estimates. First of all, set

$$g(x,t) = (1-x)f_1(t) + xf_2(t), (x,t) \in Q_T.$$

We multiply Eq.(4.8) by $w_t - g_t(x,t)$ and then integrate over $Q_T$. Using the assumption H(4.1) and the growth condition for $\sigma(x,t,u)$, we obtain, after some routine calculations, that

$$\begin{aligned}&\sup_{0\le t\le T}\int_0^1 [w_t^2 + w_x^2]dx + \int_0^T\int_0^1 \sigma(x,t,u)w_t^2 dxdt\\ &\le C_1 + C_2\int_0^T\int_0^1 |u|dxdt,\end{aligned}$$

where the constants $C_1$ and $C_2$ depend only on known data, but not on $n$.

On the other hand, we use an estimate from the paper [3] to obtain

$$\int_0^1 |u|dx \le C_2 + C_4\int_0^T\int_0^1 \sigma(x,t,u)|w_t|^2 dxdt,$$

where the constants $C_3$ and $C_4$ depend only on known data.

It follows that

$$\int_0^1 |u|dx \le C_3 + C_4[C_1 + C_2\int_0^T\int_0^1 |u|dxdt].$$

The above estimate holds if we replace $T$ by any $T' \in [0,T]$, we can apply Gronwall's inequality to obtain

$$\int_0^1 |u|dx \le C_5,$$

where $C_5$ depends only on known data.

Next, we multiply Eq. (4.9) by u and then integrate over $Q_{T'}$ with $T' \in (0,T]$. Using the same technique as in Section 2, we see

$$\begin{aligned}&\int\int_{Q_{T'}} A_n(x,u)_t u dxdt \ge b_0\int_\Omega u^2 dx - C_6,\\ &\int\int_{Q_{T'}} u_{xx}u dxdt = -\int\int_{Q_{T'}} u_x^2 dxdt\end{aligned}$$

where $b_0 > 0$ and $C_6$ depend only on known data.

It follows that

$$\int_0^1 u^2(x,T')dx + \int_0^{T'}\int_0^1 u_x^2 dxdt \le C + \int_0^T\int_0^1 |u|\sigma(x,t,u)w_t^2 dxdt$$
$$\le C + C\int_0^T ||u||^2_{L^\infty(0,1)}dt.$$

Now, by Sobolev's embedding ([10]),

$$\begin{aligned} ||u||^2_{L^\infty(0,1)} &\le C||u||_{W^{1,2}(0,1)}||u||_{L^2(0,1)} \\ &\le \delta\int_0^1 [u^2+u_x^2]dx + C(\delta)\int_0^1 u^2 dx. \end{aligned}$$

We combine the above estimates and choose $\delta$ sufficiently small to obtain

$$\int_0^1 u^2 dx + \int_0^{T'}\int_0^1 u_x^2 dxdt \le C + C\int_0^{T'}\int_0^1 u^2 dxdt.$$

Again Gronwall's inequality yields

$$\int_0^1 u^2 dx + \int_0^{T'}\int_0^1 u_x^2 dxdt \le C_8,$$

where the constant $C_8$ depends only on known data.

With those uniform estimates, as for Theorem 2.4 we can extract a subsequence from $(w_n(x,t), u_n(x,t))$ and then take the limit to obtain a weak solution to the problem (4.4)-(4.7). We shall not repeat these steps here.

Q.E.D.

**Remark 4.1**: It would be interesting to show that the temperature is continuous over $Q_T$ (see [1, 4]).

**Remark 4.2**: The uniqueness is an open question even for one-space dimension.

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