Markov Repairable Systems with History-Dependent Up and Down States

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Abstract

This paper introduces a Markov model for a multi-state repairable system in which some states are changeable in the sense that whether those states are up or down depends on the recent system evolution history. Several reliability indexes, such as availability, mean up time, steady-state reliability, are calculated using the matrix method. A sufficient condition under which the availabilities of the monotone repairable systems with history-dependent states can be compared is also obtained. Some examples are presented to illustrate the results in the paper.

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1 Introduction

A repairable system operating in a random environment is subjected to degradation, failure and various kinds of repairs. In many cases, such a system can be modeled as a multi-state Markov repairable system in which some states are classified as operational, or up states, whereas others are classified as failure, or down states, depending on the physical conditions of the system. However, in certain situations, some system states can be up and down, and their status of being up or down is affected by the recent system evolution history. The focus of this paper is on a Markov model for the multi-state repairable system with such history-dependent states.

Consider a Markov repairable system that has 8 states, say states 1, 2, 3, 4, 5, 6, 7 and 8. The states 1, 2, 3, and 4 are functional or up states, and 7 and 8 are failure or down states. The states 5, and 6 are history-dependent. The system evolution follows a continuous-time Markov chain which starts at 1. As time goes by, the system changes its states due to wear, damage, regular maintenance and repairs. When the system moves into states 5 and 6 from the up states, 5 and 6 are also up states. When the system are in one of the down states, major repairs are needed to restore the system back to the up states. But with possible imperfect repairs, the system may move into states 5 and 6 from the down states, and when this happens, the system is still in the down mode. The repair action on a failed system has to be significant so that the system is brought back to one of the up states 1, 2, 3, and 4 and then works again. For this multi-state repairable system, whether states 5 and 6 are up or down depends on the type of the last state from which the system exits before entering states 5 and 6. Figure 1 shows a possible system evolution path and the corresponding up and down durations of the system.

In general, the system dynamics of a continuously-monitored, multi-state repairable system can be described by a homogeneous, continuous-time Markov chain \( \{X(t), t \geq 0\} \), where \( X(t) \) is the system state at time \( t \). The state space \( \mathcal{S} \) is finite, and can be partitioned into three sets, \( U \), \( C \), and \( D \). The states in \( U \), also called type \( U \) states, are the up states of the system, and the states in \( D \), also called type \( D \) states, are the down states of the system. The states in \( C \), called type \( C \) or changeable states, are history-dependent, in the sense that any type \( C \) state is an up state if the last non-type \( C \) state which the chain visits prior to moving into it is of type \( U \), and a down state otherwise. As in our illustrative example above, we equip the state space \( \mathcal{S} = U \cup C \cup D \) with a partial order \( \leq \) on \( U \), \( C \), and \( D \), such that

1. if state \( a \in U \) and state \( b \in C \cup D \) are comparable, then \( a \leq b \), and
2. if state $a \in C$ and state $b \in D$ are comparable, then $a \leq b$.

The Markov chain starts, with probability 1, at an up state $i_{\min}$ that is the smallest with respect to $\leq$.

Such a multi-state Markov repairable system has a natural interpretation that the changeable set $C$ serves as a set of ‘boundary’ states between up and down states, and the system in a type $C$ state may need some preventive maintenance actions or minor repairs to get back to a good operational condition, whereas in a down state, major repairs are needed to restore the system to be operational again. In this paper, we use the matrix method to present a steady-state analysis for the system, and obtain explicit expressions for several system reliability indexes, such as availability, steady-state reliability, mean up and down times. We also compare availabilities for multi-state monotone repairable systems with different model parameters, and our results reveal some structural insights of the system dynamics with history-dependent states.

The Markov repairable systems with fixed functional and failure states have been studied in Shaked and Shanthikumar (1990), and Sumita, Shanthikumar and Masuda (1987), and the reference therein. However, the system with history-dependent states has not been studied in the reliability literature, and such a system provides a realistic model for multi-state repairable systems with actual implementations of various maintenance policies. As we will illustrate later in the paper, a system with history-dependent states can be converted into a system with fixed up and down states, by enlarging the state space, but our method is
more similar to the matrix method employed by Colquhoun and Hawkes (1982, 1990), Jalali and Hawkes (1992a, 1992b), Ball and Davies (1997), and Ball et al (1997) for studies on ion channels. This matrix method is powerful, and allows us to obtain tractable formulas of all the popular reliability indexes for the system performance.

The paper is organized as follows. Section 2 discusses the system availability. Section 3 studies the system up and down times and steady-state reliability. Section 4 presents a numerical example to illustrate the results obtained in the paper. Finally, some remarks in Section 5 conclude the paper. Throughout this paper, the terms ‘increasing’ and ‘decreasing’ mean ‘non-decreasing’ and ‘non-increasing’ respectively. A function on \( S \) is often written as a column vector of \( |S| \)-dimension, whereas a probability mass function on \( S \) is always written as a row vector of \( |S| \)-dimension. All the matrices and vectors are written in bold face, and any product of matrices and/or vectors are understood as a product with appropriate sizes.

2 System Availability and Its Comparisons

In this section, we first obtain a formula for the availability of a Markov repairable system with history-dependent states, and then compare the availabilities for the repairable systems with different model parameters. Our result shows that the availability of a monotone repairable system operating in a harsher environment is smaller.

Consider Markov chain \( \{X(t), t \geq 0\} \) we introduced in Section 1. Let \( n \) be the number of states in the state space \( S \), which has a minimal state \( i_{\min} \). First, we define an \( n \)-dimensional (row) vector, \( \mathbf{p}(t) = (p_j(t), j \in S) \), with elements given by

\[
p_j(t) = P(\text{system is in state } j \text{ at time } t) = P(X(t) = j), \; j \in S.
\] (2.1)

It is well-known that the probability vector \( \mathbf{p}(t) \) satisfies

\[
\frac{d\mathbf{p}(t)}{dt} = \mathbf{p}(t)\mathbf{Q},
\]

where \( n \times n \) matrix \( \mathbf{Q} = (q_{ij}) \) is the infinitesimal generator of the chain. Assume the system is new at time \( t = 0 \), and thus Markov chain starts at \( i_{\min} \) almost surely. Let \( \mathbf{p}_0 \) be the initial probability vector of the process.

We also define an \( n \times n \) matrix, \( \mathbf{P}(t) = (P_{ij}(t)) \), with elements given by

\[
P_{ij}(t) = P(X(t) = j \mid X(0) = i), \; i, j \in S.
\] (2.2)
It is also a standard result that

\[ \frac{dP(t)}{dt} = P(t)Q, \] with \( P(0) = I. \)

Hereafter, \( I \) denotes an identity matrix with appropriate size. Thus \( P(t) = \exp(Qt) \) and \( p(t) = \exp(Qt) \) for any \( t \geq 0. \) Let \( \tilde{\phi}(s) \) denote the Laplace transform of a function \( \phi(t), \) then the Laplace transforms of \( p(t) \) and \( P(t) \) are given by, respectively,

\[ \tilde{p}(s) = p_0(sI - Q)^{-1}, \quad \tilde{P}(s) = (sI - Q)^{-1}. \] \( (2.3) \)

To calculate explicitly the reliability indexes, it is crucial to obtain the probability that the repairable system \( \{X(t), t \geq 0\} \) remains within a specific set of states throughout the time from 0 to \( t. \) Let state space \( S = A \cup B, \) with \( A \cap B = \emptyset. \) The generator \( Q \) can now be partitioned according to \( A \) and \( B \) as follows,

\[ Q = \begin{pmatrix} Q_{AA} & Q_{AB} \\ Q_{BA} & Q_{BB} \end{pmatrix}. \] \( (2.4) \)

Consider an \( |A| \times |A| \) matrix \( A_{AA}(t) = (A P_{ij}(t)), \) where

\[ A P_{ij}(t) = P(X(t) = j, X(s) \in A \text{ for all } 0 \leq s \leq t \mid X(0) = i), \quad i, j \in A. \] \( (2.5) \)

It is known (see, for example, Colquhoun and Hawkes 1982) that

\[ \frac{dP_{AA}(t)}{dt} = P_{AA}(t)Q_{AA}, \] with initial condition \( P_{AA}(0) = I. \) Thus, \( P_{AA}(t) = \exp(Q_{AA}t) \) and its Laplace transform is given by \( \tilde{P}_{AA}(s) = (sI - Q_{AA})^{-1}. \)

Since the state space is finite, we can use the uniformization to obtain an alternative representation for \( p(t). \) Choose a real number \( q < \infty \) for the generator \( Q \) such that

\[ q \geq \max_{i \in S}\{ -q_{ii} \}. \]

Define the stochastic matrix

\[ A_q = I + \frac{1}{q}Q, \] \( (2.6) \)

and \( A_q \) is known as the probability transition matrix of the embedded, discrete-time Markov chain of \( \{X(t), t \geq 0\}. \) Substituting \( Q = -q(I - A_q) \) in \( p(t) = p_0 \exp(Qt), \) we obtain that

\[ p(t) = p_0 \left[ \sum_{k=0}^{\infty} \left( \exp(-qt) \frac{(qt)^k}{k!} \right) A_q^k \right] = \sum_{k=0}^{\infty} \left( \exp(-qt) \frac{(qt)^k}{k!} \right) p_0 A_q^k. \] \( (2.7) \)
This representation of \( \{X(t), t \geq 0\} \) is known as the discrete-time Markov chain subordinated to a Poisson process.

The instantaneous availability \( A(t) \) of a system at time \( t \) is the probability that the system is functioning at time \( t \), and its limit \( A(\infty) = \lim_{t \to \infty} A(t) \), if exists, is called the steady-state system availability.

**Theorem 2.1.** For a Markov repairable system with history-dependent states, the instantaneous availability is given by

\[
A(t) = \sum_{k \in U} p_k(t) + \sum_{i \in U} \sum_{j \in C} \sum_{l \in C} q_{ij} \int_0^t p_i(u) C P_{jl}(t - u) du,
\]

and its steady-state availability is given by

\[
A(\infty) = \lim_{s \to 0} \left[ \sum_{k \in U} s \tilde{p}_k(s) + s \sum_{i \in U} \sum_{j \in C} \sum_{l \in C} q_{ij} \tilde{p}_i(s) C \tilde{P}_{jl}(s) \right].
\]

**Proof.** The first term in the expression for \( A(t) \) is the probability that the system stays within \( U \) at time \( t \), and the second term corresponds to the probability that after having stayed in \( U \) at time \( u \), the system moves into \( C \) and stays there from \( u \) to \( t \). The system availability is the sum of these two terms. The steady-state availability then follows from Tauber theorem (Theorem 4.3, Widder 1946) and the convolution formula for the Laplace transform.

Theorem 2.1 can be used to calculate the system availability (see Section 4), but offers a little insight on how the availability would change in response to the change of system parameters. To understand this, we utilize the stochastic comparison method. Let \( X \) and \( Y \) be two \( S \)-valued random variables with \( n \) dimensional probability mass (row) vectors \( p \) and \( q \) respectively. Let \( e_A \) denote the indicator function (\( n \) dimensional column vector) of subset \( A \) of \( S \); that is, \( e_A(i) = 1 \) if \( i \in A \subseteq S \), and zero otherwise. \( X \) is said to be larger than \( Y \) in the usual stochastic order (denoted by \( X \geq_{st} Y \) or \( p \geq_{st} q \)) if

\[
pe_A \geq qe_A
\]

for all upper sets \( A \subseteq S \) (A subset \( A \subseteq S \) is called upper if \( i \in A \) and \( i \leq j \) imply that \( j \in A \)). It is easy to verify that \( X \geq_{st} Y \) if and only if \( E\phi(X) = p\phi \geq q\phi = E\phi(Y) \) for all real increasing functions (\( n \) dimensional column vectors) \( \phi : S \to \mathbb{R} \). The stochastic comparison of Markov chains involves the following notions (see Massey 1987, Li and Shaked 1994).
Definition 2.2. Let \( \{X(t), t \geq 0\} \) and \( \{X'(t), t \geq 0\} \) be two Markov chains with generators \( Q \) and \( Q' \) respectively.

1. The Markov chain \( \{X(t), t \geq 0\} \) is called stochastically monotone if there exists a \( q \) such that for any upper set \( A \subseteq S \), the function (\( n \) dimensional column vector) \( A_q e_A : S \rightarrow \mathbb{R} \) is increasing, where \( A_q \) is the stochastic matrix defined by (2.6).

2. We say \( Q \preceq_{st} Q' \) if \( Q e_A \leq Q' e_A \) component-wise for any upper set \( A \subseteq S \).

It is straightforward to verify from (2.6) that for any upper subset \( A \subseteq S \), if \( A_q e_A \) is increasing, then \( A_{q'} e_A \) is also increasing for any \( q' \geq q \). Because of (2.6), loosely speaking, a stochastically monotone Markov chain is more likely to move higher from a larger state, whereas \( Q \preceq_{st} Q' \) means that \( \{X'(t), t \geq 0\} \) (associated with \( Q' \)) is more likely to move higher from any state than \( \{X(t), t \geq 0\} \) (associated with \( Q \)) does. The following result, due to Whitt (1986) and Massey (1987), describes the stochastic comparison of Markov chains.

Theorem 2.3. Let \( \{X(t), t \geq 0\} \) and \( \{X'(t), t \geq 0\} \) be two Markov chains with the same finite state space \( S \) and same initial probability vector \( p_0 \), but different generators \( Q \) and \( Q' \) respectively. If

1. one of the processes is stochastically monotone, and

2. \( Q \preceq_{st} Q' \),

then \( X(t) \preceq_{st} X'(t) \) for any time \( t \geq 0 \).

Using this result, we can obtain the availability comparison of two Markov repairable systems. Let \( \{X(t), t \geq 0\} \) and \( \{X'(t), t \geq 0\} \) be two Markov repairable systems with history-dependent states, as described in Section 1. Let \( A(t) \) and \( A'(t) \) \( (A(\infty) \) and \( A'(\infty)) \) denote the (steady-state) availabilities of \( X(t) \) and \( X'(t) \) respectively.

Theorem 2.4. Suppose that two systems have the same finite state space \( S \) and same initial probability vector \( p_0 \), but different generators \( Q = (q_{ij}) \) and \( Q' = (q'_{ij}) \) respectively. If

1. one of the processes is stochastically monotone, and

2. \( Q \preceq_{st} Q' \),

then \( A(t) \geq A'(t) \) for any time \( t \geq 0 \), and also \( A(\infty) \geq A'(\infty) \).
Proof. The idea is to convert the repairable system with history-dependent states into a repairable system with fixed up and down states, so that the availability has a simpler expression.

Let $\mathcal{S} = \mathcal{S} \cup \mathcal{C}$ be the new state space, where $\mathcal{C}$ contains the same number of states as $C$ such that for any $i \in C$, there is correspondingly a unique $\tilde{i} \in \mathcal{C}$. Extend the partial order $\leq$ in $\mathcal{S}$ to $\tilde{\mathcal{S}}$ as follows.

1. Retain all the ordering relations among the states in $\mathcal{S} = U \cup C \cup D$.
2. For any $\tilde{i} \in \mathcal{C}$ and $j \in D$, if $i \leq j$ in $\mathcal{S}$, then define $\tilde{i} \leq \tilde{j}$ in $\tilde{\mathcal{S}}$.
3. For any $i \in U$ and $\tilde{j} \in \mathcal{C}$, if $i \leq j$ in $\mathcal{S}$, then define $i \leq \tilde{j}$ in $\tilde{\mathcal{S}}$.
4. For any $\tilde{i}, \tilde{j} \in \mathcal{C}$, if $i \leq j$ in $\mathcal{S}$, then define $\tilde{i} \leq \tilde{j}$ and $i \leq \tilde{j}$ in $\tilde{\mathcal{S}}$.

Note that for any state $i \in C$, $i \leq \tilde{i}$ in $\tilde{\mathcal{S}}$, but a state in $C$ is not larger than any state in $\mathcal{C}$.

It is straightforward to verify that $\tilde{\mathcal{S}}$ is a partially ordered space with this extension of $\leq$.

Let $\{Y(t), t \geq 0\}$ and $\{Y'(t), t \geq 0\}$ be two Markov chains with state space $\tilde{\mathcal{S}}$, starting at $i_{\text{min}}$ almost surely. The generator $Q = (q_{ij})$ $(Q' = (q'_{ij}))$ of $Y(t)$ $(Y'(t))$ is defined as follows.

$$
q_{ij} \quad (q'_{ij}) = \begin{cases} 
q_{ij} \quad (q'_{ij}) & \text{if } i \in U \cup C \cup D, \text{ and } j \in U \cup D \\
q_{kj} \quad (q'_{kj}) & \text{if } i = \tilde{k} \in \tilde{C}, \text{ and } j \in U \cup D \\
q_{ij} \quad (q'_{ij}) & \text{if } i \in C, \text{ and } j \in C \\
q_{kl} \quad (q'_{kl}) & \text{if } i = \tilde{k} \in \tilde{C}, \text{ and } j = \tilde{l} \in \tilde{C} \\
q_{ij} \quad (q'_{ij}) & \text{if } i \in U, \text{ and } j \in C \\
q_{ik} \quad (q'_{ik}) & \text{if } i \in D, \text{ and } j = \tilde{k} \in \tilde{C} \\
0 \quad (0) & \text{otherwise.}
\end{cases}
$$

Note that the new Markov chains can not move directly from $U \cup C$ to $\tilde{C}$, nor from $D \cup \tilde{C}$ to $C$ (Figure 2). Obviously $A(t) = P(Y(t) \in U \cup C)$ and $A'(t) = P(Y'(t) \in U \cup C)$.

In order to use Theorem 2.3 to compare the processes, we need to show that the sufficient conditions in Theorem 2.3 hold for $\{Y(t), t \geq 0\}$ and $\{Y'(t), t \geq 0\}$. We only prove the case where $\{X(t), t \geq 0\}$ is stochastically monotone, and the other case is similar. Let $A_q = (a_{ij})$ and $\tilde{A}_q = (\tilde{a}_{ij})$ be the stochastic matrices for $X(t)$ and $Y(t)$, respectively, defined as in (2.6).
From (2.9), we have,

\[
\bar{a}_{ij} = \begin{cases} 
  a_{ij} & \text{if } i \in U \cup C \cup D, \text{ and } j \in U \cup D \\
  a_{k_j} & \text{if } i = \bar{k} \in \bar{C}, \text{ and } j \in U \cup D \\
  a_{ij} & \text{if } i \in \bar{C}, \text{ and } j \in \bar{C} \\
  a_{kl} & \text{if } i = \bar{k} \in \bar{C}, \text{ and } j = \bar{l} \in \bar{C} \\
  a_{ij} & \text{if } i \in U, \text{ and } j \in \bar{C} \\
  a_{ik} & \text{if } i \in D, \text{ and } j = \bar{k} \in \bar{C} \\
  0 & \text{otherwise.} 
\end{cases}
\] (2.10)

The monotonicity of \(\{X(t), t \geq 0\}\) implies that \(A_q e_A\) is increasing for any upper set \(A \subseteq S\).

For any upper subset \(A \subseteq \mathcal{S}\), let

\[
\begin{align*}
\bar{B} &= A \cap (U \cup \bar{C} \cup D) = (A \cap U) \cup (A \cap \bar{C}) \cup (A \cap D), \\
B &= (A \cap U) \cup \{i \in C : \bar{i} \in A \cap \bar{C}\} \cup (A \cap D).
\end{align*}
\]

Since \(A\) is upper, \(A \cap S\) is an upper subset of \(S\), and also \(\bar{B}\) is an upper subset of \(U \cup \bar{C} \cup D\). We claim that \(B\) is an upper subset of \(S\). For this, consider \(i \in B\), and \(i \leq j\) for \(j \in S\). We need to show that \(j \in B\).

1. Suppose that \(i \in A \cap (U \cup D)\). Because \(A\) is upper, \(j \in A\) and \(j \in A\).
(a) If \( j \in U \cup D \), then \( j \in A \cap (U \cup D) \subseteq B \).

(b) If \( j \in C \), then \( \bar{j} \in A \cap \bar{C} \), which implies that \( j \in \{ i \in C : \bar{i} \in A \cap \bar{C} \} \subseteq B \).

2. Suppose that \( i \in C \) such that \( \bar{i} \in A \cap \bar{C} \).

(a) If \( j \in C \), then \( \bar{i} \leq \bar{j} \) and so \( j \in A \). Thus \( \bar{j} \in A \cap \bar{C} \), which implies that \( j \in \{ i \in C : \bar{i} \in A \cap \bar{C} \} \subseteq B \).

(b) If \( j \in D \), then \( \bar{i} \leq j \) and so \( j \in A \). Thus \( j \in A \cap D \subseteq B \).

Therefore, in any case, \( j \in B \). Indeed, \( B \) is upper.

We also claim that \( A \cap S \subseteq B \). In fact, for any \( j \in A \cap C \), we have \( j \leq \bar{j} \in C \). Since \( A \) is upper, \( \bar{j} \in A \). Thus \( \bar{j} \in A \cap \bar{C} \), which implies that \( j \in \{ i \in C : \bar{i} \in A \cap \bar{C} \} \). Hence \( A \cap C \subseteq \{ i \in C : \bar{i} \in A \cap \bar{C} \} \), and \( A \cap S \subseteq B \).

We first show that \( \{ Y(t), t \geq 0 \} \) is stochastically monotone. We need to verify that \( \bar{A}_q e_A \) is increasing for any given upper subset \( A \subseteq \bar{S} \). Let \( a_i \) (\( \bar{a}_i \)) denote the \( i \)th row vector of \( A_q \) (\( \bar{A}_q \)). From our construction, there are three cases.

1. For any \( i \in U \cup C \), since \( \{ Y(t), t \geq 0 \} \) can not move directly from \( i \) into \( \bar{C} \) (see (2.10)),
   \[ a_i e_A = \bar{a}_i e_A \subseteq S. \]
   Hence \( a_i e_A \) is increasing in \( i \) within \( U \cup C \).

2. For any \( i \in D \cup \bar{C} \), \( \{ Y(t), t \geq 0 \} \) can not move directly from \( i \) into \( C \) (2.10).

   (a) If \( i \in D \), then \( a_i e_A = \bar{a}_i e_B = a_i e_B \). Hence \( a_i e_A \) is increasing in \( i \) within \( D \).

   (b) If \( i = \bar{k} \in \bar{C} \), then \( a_i e_A = \bar{a}_i e_B = a_k e_B \). Hence \( a_i e_A \) is increasing in \( i \) within \( \bar{C} \).

   (c) Consider \( i = \bar{k} \in \bar{C} \) and \( j \in D \) with \( i \leq j \). From our construction, we have \( k \leq j \)
   and thus,
   \[ a_i e_A = \bar{a}_i e_B = a_k e_B \leq a_j e_B = \bar{a}_j e_B = \bar{a}_j e_A. \]
   Therefore, \( a_i e_A \) is increasing in \( i \) within \( D \cup \bar{C} \).

3. Consider any \( i \in U \cup C \), and \( j \in D \cup \bar{C} \) with \( i \leq j \).

   (a) If \( j \in D \), then
   \[ a_i e_A = a_i e_{A \cap S} \leq a_j e_{A \cap S} \leq a_j e_B = \bar{a}_j e_A. \]

   (b) If \( j \in \bar{C} \), then \( j = \bar{k} \) for some \( k \in C \). From our construction of the partial ordering, \( i \leq \bar{k} \) is equivalent to \( i \leq k \). Thus,
   \[ a_i e_A = a_i e_{A \cap S} \leq a_k e_{A \cap S} \leq a_k e_B = \bar{a}_k e_B = \bar{a}_j e_A. \]
Therefore, $a_i e_A$ is increasing in $i$ for any upper subset $A \subseteq \hat{S}$.

Next, we show that $Q \leq_{st} Q'$. Let $q_i$ ($\bar{q}_i$) denote the $i$th row vector of $Q$ ($\bar{Q}$), and $q'_i$ ($\bar{q}'_i$) denote the $i$th row vector of $Q'$ ($\bar{Q}'$). We now verify that for any upper set $A \subseteq \hat{S}$, $\bar{q}_i e_A \leq \bar{q}'_i e_A$ for any $i \in \hat{S}$.

1. For any $i \in U \cup C$,
   $$\bar{q}_i e_A = q_i e_{A \cap S} \leq q'_i e_{A \cap S} = \bar{q}'_i e_A.$$

2. For any $i \in D$,
   $$q_i e_A = q_i e_B = q_i e_B \leq q'_i e_B = \bar{q}'_i e_A.$$

3. For any $i = \bar{k} \in C$,
   $$\bar{q}_i e_A = q_k e_B = q_k e_B \leq q'_k e_B = \bar{q}'_i e_A.$$

Thus $\bar{Q} e_A \leq \bar{Q}' e_A$ component-wise for any upper set $A$.

From Theorem 2.3, we obtain that $Y(t) \leq_{st} Y'(t)$ for any $t$. Since $D \cup \bar{C}$ is an upper subset in $\hat{S}$, then $P(Y(t) \in D \cup \bar{C}) \leq P(Y'(t) \in D \cup \bar{C})$, and hence $A(t) \geq A'(t)$ for any $t$. Taking limit leads to $A(\infty) \geq A'(\infty)$.  

Our interpretation for $Q \leq_{st} Q'$ is that the system $X'(t)$ receives severer wear and damage than $X(t)$ does, and thus, the availability of $X'(t)$ is smaller.

The technique used in Theorem 2.4 can also be used for the availability calculation. Since the new state space $\hat{S}$ has fixed up and down states, then

$$A(t) = P(Y(t) \in U \cup C) = p_0 \exp(Q t) e_{U \cup C}.$$  \hspace{1cm} (2.11)

However, enlarging state space may add extra burden on computation.

**Example 2.5.** Consider a continuous-time Markov repairable system $\{X(t), t \geq 0\}$ with state space $S = \{1, 2, 3\}$, where 1 is the up state, 3 is the down state, and 2 is changeable. The state space is equipped with the natural ordering. The Markov chain starts at 1, and has the following generator.

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$
Let \( q = 3 \), then

\[
A_q = \begin{pmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{pmatrix}.
\]

Thus \( \{X(t), t \geq 0\} \) is stochastically monotone. Let \( \{X'(t), t \geq 0\} \) be another Markov repairable system with the same state space and same initial probability vector, and the following generator.

\[
Q' = \begin{pmatrix}
-2 & \frac{1}{2} & \frac{3}{2} \\
\frac{1}{2} & -2 & \frac{3}{2} \\
\frac{1}{2} & \frac{3}{2} & -2
\end{pmatrix}.
\]

It is easy to verify that \( Q \leq_{st} Q' \), and hence \( A(t) \geq A'(t) \) for any \( t \).

To calculate the availability using (2.11), we enlarge the state space and let \( \mathcal{S} = \{1, 2, 2, 3\} \) with the ordering that \( 1 \leq 2 \leq 2 \leq 3 \). The corresponding generators on this enlarged state space are given by (see (2.9))

\[
\bar{Q} = \begin{pmatrix}
-2 & 1 & 0 & 1 \\
1 & -2 & 0 & 1 \\
1 & 0 & -2 & 1 \\
1 & 0 & 1 & -2
\end{pmatrix}, \quad \text{and} \quad \bar{Q}' = \begin{pmatrix}
-2 & \frac{1}{2} & 0 & \frac{3}{2} \\
\frac{1}{2} & -2 & 0 & \frac{3}{2} \\
\frac{1}{2} & 0 & -2 & \frac{3}{2} \\
\frac{1}{2} & 0 & \frac{3}{2} & -2
\end{pmatrix}.
\]

It follows from (2.11) that the availability of \( X(t) \) is given by \( A(t) = (1, 0, 0, 0) \exp(\bar{Q}t)\text{e}_{\{1,2\}} \), and the availability of \( X'(t) \) is given by \( A'(t) = (1, 0, 0, 0) \exp(\bar{Q}'t)\text{e}_{\{1,2\}} \).

3 Steady-State Analysis of System Up and Down Times

To calculate the distributions of up and down times for a Markov repairable system \( \{X(t), t \geq 0\} \) with history-dependent states, a useful quantity is the probability that the system stays within a subset \( A \) of state space \( \mathcal{S} \) up to time \( t \) and then exits from \( A \) to a state outside. Let state space \( \mathcal{S} = A \cup B \). Consider,

\[
g_{ij}^{AB}(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(X(s) \in A \text{ for all } 0 \leq s \leq t, X(t + \Delta t) = j \mid X(0) = i), \quad i \in A, j \in B.
\]

It follows from Colquhoun and Hawkes (1982) that

\[
g_{ij}^{AB}(t) = \sum_{r \in A} A P_{ir}(t) q_{rj}, \quad i \in A, j \in B, \tag{3.1}
\]
where \( ^4P_{rr}(t) \) is defined by (2.5). In the matrix form, we obtain an \(|A| \times |B|\) matrix,

\[
G_{AB}(t) = (g_{ij}^{AB}(t)) = P_{AA}(t)Q_{AB},
\]

where \( Q_{AB} \) is given in the partitioned matrix (2.4). Its Laplace transform is given by

\[
\tilde{G}_{AB}(s) = (sI - Q_{AA})^{-1}Q_{AB}.\]

In particular, \( \tilde{G}_{AB}(0) = (\tilde{g}_{ij}^{AB}(0)) = P_{AA}(0)Q_{AB} \), where

\[
\tilde{g}_{ij}^{AB}(0) = \int_0^{\infty} g_{ij}^{AB}(u)du = P(\text{system exits from } A \text{ to } j \mid X(0) = i), i \in A, j \in B.
\]

Let

\[
G_{AB} = -Q_{AA}^{-1}Q_{AB}
\]

denote the matrix of exit probabilities. Note that \( g_{ij}^{AB}(t), i \in A, j \in B, \) is not a proper probability density function.

We now partition the generator of Markov chain \( \{X(t), t \geq 0\} \) according to the types of states we defined in Section 1. Let

\[
Q = \begin{pmatrix}
Q_{UU} & Q_{UC} & Q_{UD} \\
Q_{CU} & Q_{CC} & Q_{CD} \\
Q_{DU} & Q_{DC} & Q_{DD}
\end{pmatrix}.
\]

(3.3)

Suppose that the system is in the steady-state. An up duration for the repairable system begins when a transition from \( D \) directly to \( U \) (\( D \to U \)), or from \( D \), via \( C \), to \( U \) (\( D \to C \to U \)) occurs. The rate that the system goes into the up state \( j \) is given by

\[
\sum_{i \in D} \left[ p_i(\infty)(q_{ij} + \sum_{k \in C} q_{ik} \tilde{g}_{kj}^{CU}(0)) \right], \quad j \in U,
\]

where \( p_i(\infty) \) denotes the equilibrium probability of the system in state \( i \). Let \( u_U = (u_j, j \in U) \) be the vector of probability masses that the system goes into the up states, then

\[
u_U = \frac{p_D(\infty)(Q_{DU} + Q_{DC}G_{CU})}{p_D(\infty)(Q_{DU} + Q_{DC}G_{CU})e}
\]

(3.4)

where \( p_D(\infty) = (p_i(\infty), i \in D) \), and hereafter, \( e \) denotes the vector of 1’s with an appropriate dimension.

**Theorem 3.1.** Let \( \{X(t), t \geq 0\} \) be a Markov repairable system with history-dependent states, as described in Section 1. The probability density function of the lifetime of the system in steady-state is given by

\[
f_{up}(t) = (u_U, 0) G_{WD}(t)e,
\]

where \( W = U \cup C \). The mean up time for the repairable system in the steady-state is

\[
m_{up} = (u_U, 0)Q_{WW}^{-2}Q_{WD}e.
\]
Proof. The expression of $f_{up}$ follows from the fact that the system up time starts at the transition $D \rightarrow U$ or $D \rightarrow C \rightarrow U$, and ends when a transition from $W$ to $D$ occurs. Since the exit from $W$ is certain, we have

$$\int_0^\infty (u_U, 0)G_{WD}(t)e\ dt = P(\text{system exits from } W \mid \text{system starts at an up state}) = 1.$$  

Thus, $(u_U, 0)G_{WD}(t)e$ is a proper density function.

Since $G_{WD}(t) = P_{WW}(t)Q_{WD}$, its Laplace transform is given by $\tilde{G}_{WD}(s) = (sI - Q_{WW})^{-1}Q_{WD}$. Thus, $\tilde{f}_{up}(s) = (u_U, 0)(sI - Q_{WW})^{-1}Q_{WD}e$. This implies that

$$m_{up} = -\left( \frac{df_{up}(s)}{ds} \right)_{s=0} = (u_U, 0)Q_{WW}^{-2}Q_{WD}e. $$

The steady-state reliability of the system can be also calculated as follows,

$$R_\infty(t) = \int_t^\infty f_{up}(u)du = -(u_U, 0)e^{-Q_{WW}t}Q_{WW}^{-1}Q_{WD}e.$$  

Similarly, the distribution of the system down time in the steady-state can be also obtained.

Theorem 3.2. Let $\{X(t), t \geq 0\}$ be a Markov repairable system with history-dependent states, as described in Section 1. The probability density function of the downtime of the system in steady-state is given by

$$f_{down}(t) = (u_D, 0)G_{FU}(t)e,$$

where $F = D \cup C$, and

$$u_D = \frac{p_U(\infty)(Q_{UD} + Q_{UC}G_{CD})}{p_U(\infty)(Q_{UD} + Q_{UC}G_{CD})}$$

is the vector of probabilities that the repairable system goes into the down states. The mean down time for the repairable system in the steady-state is $m_{down} = (u_D, 0)Q_{FD}^{-2}Q_{FU}e$.

Note that since the exit from $F$ is certain, $f_{down}(t)$ is a proper probability density function.

As we mentioned before, the system in a type $U$ state operates in a good condition, whereas the system in a type $D$ state needs a major repair to get back to working condition. The system in a type $C$ state may be functional in some cases, but may need some minor repairs to restore it to a good operational condition. It is then of interest to calculate the total time that the system operates in a good condition per major repair cycle; that is, the
total time that the system spends in $U$ within the set of up states before moving to a down state.

Let $\{X(t), t \geq 0\}$ be a Markov repairable system with history-dependent states, as described in Section 1. Let $f_{\text{good}}(t)$ be the probability density function of the total time that the system, in the steady-state, spends in $U$ per major repair cycle. To calculate $f_{\text{good}}(t)$, we use the method employed in Colquhoun and Hawkes (1982). First, we observe that the Laplace transform of the sojourn time that the system last visits $U$ before moving to a down state is given by

$$\tilde{G}_{UC}(s)G_{CD} + \tilde{G}_{UD}(s)e_D.$$ 

Given that there are $k$ visits to $U$ before going to a down state, the sum of the first $k - 1$ occupation times in $U$ has the following Laplace transform,

$$u_U[\tilde{G}_{UC}(s)G_{CU}]^{k-1},$$

where $u_U$ is the vector of probabilities that the system goes into an up state in the steady-state. Thus, the Laplace transform of $f_{\text{good}}(t)$ is given by,

$$\tilde{f}_{\text{good}}(s) = \sum_{k=1}^{\infty} u_U[\tilde{G}_{UC}(s)G_{CU}]^{k-1}[\tilde{G}_{UC}(s)G_{CD} + \tilde{G}_{UD}(s)]e_D$$

$$= u_U[I - \tilde{G}_{UC}(s)G_{CU}]^{-1}[\tilde{G}_{UC}(s)G_{CD} + \tilde{G}_{UD}(s)]e_D.$$ 

Since $\tilde{G}_{UC}(s) = (sI - Q_{UU})^{-1}Q_{UC}$ and $\tilde{G}_{UD}(s) = (sI - Q_{UU})^{-1}Q_{UD}$, we have

$$\tilde{f}_{\text{good}}(s) = u_U(I - (sI - Q_{UU})^{-1}Q_{UC}G_{CU})^{-1}(sI - Q_{UU})^{-1}[Q_{UC}G_{CD} + Q_{UD}]e_D$$

$$= u_U[(sI - Q_{UU})(I - (sI - Q_{UU})^{-1}Q_{UC}G_{CU})]^{-1}[Q_{UC}G_{CD} + Q_{UD}]e_D$$

$$= u_U[sI - Q_{UU} - Q_{UC}G_{CU}]^{-1}[Q_{UC}G_{CD} + Q_{UD}]e_D.$$ 

Noticing that the Laplace transform of $\exp[(Q_{UU} + Q_{UC}G_{CU})t]$ is $[sI - Q_{UU} - Q_{UC}G_{CU}]^{-1}$, we then obtain the density function $f_{\text{good}}$. This and related results are summarized in the following theorem.

**Theorem 3.3.** Consider a Markov repairable system with history-dependent states.

1. The probability density function of the total time that the system operates in a good condition (that is, in type $U$ states) per major repair cycle is given by

$$f_{\text{good}}(t) = u_U \left( \exp[(Q_{UU} + Q_{UC}G_{CU})t] \right) [Q_{UC}G_{CD} + Q_{UD}]e_D.$$  

(3.5)
The mean time $m_{good}$ that the system operates in a good condition per major repair cycle is given by

$$m_{good} = u_U(Q_{UU} + Q_{UC}G_{CU})^{-2}[Q_{UC}G_{CD} + Q_{UD}]e_D. \quad (3.6)$$

2. The probability density function of the total time that the system stays in an overhaul mode (that is, in type $D$ states) per down cycle is given by

$$f_{bad}(t) = u_D (\exp[(Q_{DD} + Q_{DC}G_{CD})t]) [Q_{DC}G_{CU} + Q_{DU}]e_U. \quad (3.7)$$

The mean time $m_{bad}$ that the system stays in an overhaul mode per down cycle is given by

$$m_{bad} = u_D(Q_{DD} + Q_{DC}G_{CD})^{-2}[Q_{DC}G_{CU} + Q_{DU}]e_U. \quad (3.8)$$

In fact, our Markov repairable system with history-dependent states is similar to the stochastic model of ion channel developed by Colquhoun and Hawkes (1982). It is thus not surprising that the matrix method of Colquhoun and Hawkes (1982) can be used to calculate the reliability indexes of Markov repairable systems. Note, however, that the difference of our model and the ion channel model of Colquhoun and Hawkes (1982) is that some states in our repairable system are of changeable types.

### 4 A Numerical Example

In this section, we present an example to illustrate the results we obtained in the previous sections. Consider a repairable system $\{X(t), t \geq 0\}$ with 6 states, $S = \{1, 2, 3, 4, 5, 6\}$. Let $U = \{1, 2\}$, $C = \{3, 4\}$, $D = \{5, 6\}$.

Its generator is partitioned as follows.

$$Q = \begin{pmatrix} Q_{UU} & Q_{UC} & Q_{UD} \\ Q_{CU} & Q_{CC} & Q_{CD} \\ Q_{DU} & Q_{DC} & Q_{DD} \end{pmatrix} = \begin{pmatrix} -10 & 2 & 4 & 1 & 3 & 0 \\ 3 & -10 & 2 & 2 & 1 & 2 \\ 2 & 3 & -12 & 4 & 2 & 1 \\ 0 & 2 & 2 & -9 & 3 & 2 \\ 1 & 2 & 3 & 3 & -10 & 1 \\ 0 & 1 & 2 & 3 & 2 & -8 \end{pmatrix}.$$ 

By solving $p(\infty)Q = 0$ with $p(\infty)e = 1$, we obtain that

$$p_1(\infty) = \frac{7199}{70030}, \quad p_2(\infty) = \frac{5912}{35015}, \quad p_5(\infty) = \frac{6323}{35015}, \quad p_6(\infty) = \frac{5054}{35015}.$$
On the other hand,

\[ G_{CU} = -Q_{CC}^{-1}Q_{CU} = \begin{pmatrix} \frac{9}{50} & \frac{7}{20} \\ \frac{1}{25} & \frac{3}{10} \end{pmatrix}, \]

\[ u_U = \frac{p_D(\infty)(Q_{DU} + Q_{DC}G_{CU})}{p_D(\infty)(Q_{DU} + Q_{DC}G_{CU})e} = (0.25318, 0.74682). \]

After some matrix manipulations, we obtain, by using Maple software, that

\[ f_{up}(t) = 4.00763 \exp(-3.61294t) + 0.627826 \exp(-12.54363t) - 1.59274 \exp(-10.18498t) - 0.042716 \exp(-14.65845t). \]

The mean up time for the repairable system is \( m_{up} = 0.29546. \)

Similarly, we obtain that

\[ u_D = \frac{p_U(\infty)(Q_{UD} + Q_{UC}G_{CD})}{p_U(\infty)(Q_{UD} + Q_{UC}G_{CD})e} = (0.64145, 0.35855), \]

\[ f_{down}(t) = 0.40770 \exp(-9.81665t) + 2.31011 \exp(-2.63310t) + 1.20653 \exp(-13.27512t) \cos(0.61549t) - 2.74176 \exp(-13.27152t) \sin(0.61549t). \]

The mean downtime for the repairable system is \( m_{down} = 0.34279. \)

After taking the Laplace transform and the inverse transform, we obtain that

\[ A(t) = p_1(t) + p_2(t) + \sum_{i=1}^{2} \sum_{j=3}^{4} \sum_{l=3}^{4} q_{ij} \int_0^t p_i(u) C_{jl}(t - u) du, \]

where

\[ p_1(t) = 0.102799 + 0.19304 \exp(-14.23372t) + 0.00864 \exp(-13.23422t) + 0.33446 \exp(-11.72110t) \cos(0.69196t) - 0.42893 \exp(-11.72110t) \sin(0.69196t) + 0.36106 \exp(-8.08986t), \]

\[ p_2(t) = 0.16884 + 0.23728 \exp(-14.23372t) + 0.00236 \exp(-13.23422t) - 0.55405 \exp(-11.72110t) \cos(0.69196t) + 0.13315 \exp(-11.72110t) \sin(0.69196t) + 0.14557 \exp(-8.08986t), \]

\[ q_{13} = 4, \quad q_{14} = 1, \quad q_{23} = 2, \quad q_{24} = 2, \]
\[ C_{P33}(t) = -\frac{3}{41} \exp(-\frac{21}{2}t)\sqrt{41} \sinh(\frac{\sqrt{41}}{2}t) + \exp(-\frac{21}{2}t)\cosh(\frac{\sqrt{41}}{2}t), \]
\[ C_{P34}(t) = \frac{4}{41} \sqrt{41} \exp(\frac{\sqrt{41} - 21}{2}t) - \exp(-\frac{\sqrt{41} + 21}{2}t), \]
\[ C_{P43}(t) = \frac{2}{41} \sqrt{41} \exp(\frac{\sqrt{41} - 21}{2}t) - \exp(-\frac{\sqrt{41} + 21}{2}t), \]
\[ C_{P44}(t) = \frac{3}{41} \exp(-\frac{21}{2}t)\sqrt{41} \sinh(\frac{\sqrt{41}}{2}t) + \exp(-\frac{21}{2}t)\cosh(\frac{\sqrt{41}}{2}t). \]

Here, sinh(x) and cosh(x) are the hyperbolic functions. The curve of the availability \( A(t) \) is given in the top graph of Figure 3. The segment of the availability curve is magnified in the bottom graph of Figure 3. It is easy to see the availability is stabilized and the steady-state availability \( A(\infty) \approx 0.3788 \).

5 Conclusions

In this paper, we introduce a new Markov maintenance model to study the situation where a repairable system experiences certain modes in which the system behavior depends on the recent system evolution history. Such a system provides a realistic, tractable model for multi-state repairable systems with actual implementations of various maintenance policies. We employ the matrix method developed in the ion channel theory to calculate the system reliability measures, such as availability, and the distributions of up and down times. We also develop a stochastic comparison method to compare the availabilities of two systems with different parameters, and show that the availability of a monotone repairable system is reduced in a tougher operating environment. The stochastic availability comparison, such as the one presented in this paper, examines how the system availability varies in response to the environmental change. Such studies, to the best of our knowledge, have not appeared in the reliability literature.
Figure 3: *Availability Curves*: The top graph shows the availability curve, and the bottom graph details the segment of the curve from time instants 1.8 to 3.
References


