Multivariate Risk Model of Phase Type

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Abstract

This paper is concerned with several types of ruin probabilities for a multivariate compound Poisson risk model, where the claim size vector follows a multivariate phase type distribution. First, an explicit representation for the convolution of a multivariate phase type distribution is derived, and then an explicit formula for the ruin probability that the total claim surplus exceeds the total initial reserve in infinite horizon is obtained. Furthermore, the effect of the dependence among various types of claims on this type of ruin probability is considered under the convex and supermodular orders. In addition, the bounds for other types of ruin probabilities are developed by utilizing the association of multivariate phase type distributions. Finally, some examples are presented to illustrate the results.

Key words and phrases: Multivariate risk model, ruin probability, multivariate phase type distribution, Marshall-Olkin distribution, association, supermodular comparison, convex comparison.

Subject and insurance branch codes: IM11, IM13.
1 Introduction

An unexpected claim event usually triggers several types of claims in an umbrella insurance policy, where the claim sizes are often correlated. With few exceptions, however, most studies in risk/ruin theory focus on the models with independent claims. In this paper, we develop a multivariate risk model in which different types of claims are stochastically dependent, and their joint distribution is of phase type. We study several types of ruin probabilities for the multivariate risk process, and discuss the impact of dependence among claims of various types on these ruin probabilities.

The univariate classical risk process can be described as
\[ \sum_{n=1}^{N(t)} X_n - pt - u, \quad t \geq 0, \]
where \( \{N(t), t \geq 0\} \) is the claim number (counting) process, and \( X_n \) is the \( n \)th claim size, \( p > 0 \) is the premium rate and \( u \geq 0 \) is the initial reserve. For the classical compound Poisson risk model, we assume that \( \{N(t), t \geq 0\} \) is a Poisson process with rate \( \lambda \), and \( X_n \)'s are independent and identically distributed (i.i.d.) non-negative random variables and also independent of \( \{N(t), t \geq 0\} \). The ruin probability \( \psi(u) \) in infinite horizon is defined by
\[ \psi(u) = P \left( \sup_{0 \leq t < \infty} S(t) > u \right), \quad (1.1) \]
where \( S(t) = \sum_{n=1}^{N(t)} X_n - pt \) denotes the claim surplus at time \( t \). Ruin theory for the univariate risk model has been discussed extensively in the literature, and many results are summarized in Asmussen (2000) and Rolski et al. (1999).

Even in the univariate classical risk model, it is difficult to obtain explicit formulas for the ruin probability \( \psi(u) \). However, for the exponential claim sizes with common mean \( 1/\beta \), one can calculate \( \psi(u) \) explicitly,
\[ \psi(u) = \frac{1}{1 + \theta} \exp \left( -\frac{\theta \beta}{1 + \theta} u \right), \quad (1.2) \]
where \( \theta = p \beta / \lambda - 1 > 0 \) is known as the relative security loading parameter. More generally, Asmussen and Rolski (1991) gave an explicit formula of \( \psi(u) \) for the compound Poisson risk model when the claim size is of phase type in the sense of Neuts (1981). A non-negative random variable \( X \) is said to be of phase type with representation \( (\alpha, T, d) \) if \( X \) is the time to absorption into the absorbing state \( 0 \) in a finite Markov chain with state space \( \{0, 1, \ldots, d\} \) and initial distribution \( (0, \alpha) \), and infinitesimal generator,
\[
\begin{bmatrix}
0 & 0 \\
-Te & T
\end{bmatrix},
\]
where \( \mathbf{0} \) is the row vector of zeros of \( d \) dimension, and \( \mathbf{e} \) is the column vector of 1’s, and \( T \) is a \( d \times d \) matrix. The phase type distributions enjoy many desirable properties, which can be found in Neuts (1981). Utilizing the fact that \( \psi(u) \) in (1.1) is the tail probability of the stationary waiting time in the \( M/G/1 \) queue, Asmussen and Rolski (1991) showed that if the claim size in the compound Poisson risk model with a positive relative security loading parameter is of phase type with representation \((\alpha, T, d)\), then for any \( u \geq 0 \),

\[
\psi(u) = -\frac{\lambda}{p} \alpha T^{-1} \exp \left\{ \left( T - \frac{\lambda}{p} t_0 \alpha T^{-1} \right) u \right\} \mathbf{e},
\]

where \( t_0 = -T \mathbf{e} \). Because any distribution on \([0, \infty)\) can be approximated by phase type distributions, (1.3) is versatile in application.

In the univariate classical risk model, one claim event yields one claim. Recently, multivariate risk models have been also introduced and studied in the literature (see Chan et al. (2003) and the references therein). One of such multivariate risk models is motivated by an insurance portfolio discussed in Sundt (1999), which consists of \( s \) insurance policies or subportfolios. The number of claim events up to time \( t \) is described by \( N(t) \), but each claim event has effects on the \( s \) policies such as a windstorm or a vehicle accident. The claim amount under policy \( j \) caused by the \( n \)-th claim event is denoted by \( X_{j,n} \). Assume that \( u_j \geq 0 \) is the initial capital in subportfolio \( j \) or for type \( j \) claim, \( p_j > 0 \) is the premium rate in subportfolio \( j \) or for type \( j \) claim, \( j = 1, \ldots, s \). Thus, it is of interest to study the following multivariate risk process,

\[
U(t) = \begin{pmatrix}
U_1(t) \\
\vdots \\
U_s(t)
\end{pmatrix} = \begin{pmatrix}
\sum_{n=1}^{N_1(t)} X_{1,n} - p_1 t - u_1 \\
\vdots \\
\sum_{n=1}^{N_s(t)} X_{s,n} - p_s t - u_s
\end{pmatrix}, \quad t \geq 0.
\]

We assume throughout that \( \{N(t), t \geq 0\} \) is a Poisson process with rate \( \lambda \), and \( \{(X_{1,n}, \ldots, X_{s,n}), n \geq 1\} \) is a sequence of i.i.d. non-negative random vectors, and also independent of \( \{N(t), t \geq 0\} \). Sundt (1999) studied a recursive approach for the evaluation of the distribution of the multivariate aggregate claim process \( \left( \sum_{n=1}^{N_1(t)} X_{1,n}, \ldots, \sum_{n=1}^{N_s(t)} X_{s,n} \right) \). Chan et al. (2003) considered the multivariate risk process \( U(t) \) when \( s = 2 \). In both these papers, the claim sizes \( X_{1,n}, \ldots, X_{s,n} \) are assumed to be independent for any \( n \geq 1 \). In many situations, however, claims of various types \( X_{1,n}, \ldots, X_{s,n} \) are dependent since these claims are caused by the same event occurring randomly in a common environment. We assume that the relative security loading of type \( j \), \( \theta_j = p_j / (\lambda E(X_{j,n})) - 1 > 0 \), for any \( 1 \leq j \leq s \). Further, we denote the claim surplus in subportfolio \( j \) at time \( t \) by \( S_j(t) = \sum_{n=1}^{N_j(t)} X_{j,n} - p_j t, \quad t \geq 0, j = 1, \ldots, s \).
There are several types of ruin probabilities for the multivariate risk model. In this paper, we are interested in the following three ruin probabilities,
\[
\psi_{\text{sum}}(u) = P \left( \sup_{0 \leq t < \infty} \left\{ \sum_{j=1}^{s} S_j(t) > u \right\} \right),
\]
where \( u = \sum_{j=1}^{s} u_j \),
\[
\psi_{\text{or}}(u_1, \ldots, u_s) = P \left( \bigcup_{j=1}^{s} \left\{ \sup_{0 \leq t < \infty} (S_j(t)) > u_j \right\} \right),
\]
and
\[
\psi_{\text{and}}(u_1, \ldots, u_s) = P \left( \bigcap_{j=1}^{s} \left\{ \sup_{0 \leq t < \infty} (S_j(t)) > u_j \right\} \right).
\]
The ruin probability in (1.4) represents the probability that the total claim surplus of all the subportfolios in a portfolio exceeds the total initial reserve at some time; the ruin probability in (1.5) means the probability that ruin occurs in at least one subportfolio; and the ruin probability in (1.6) denotes the probability that ruin occurs, not necessarily at the same time, in all subportfolios eventually.

This paper focuses on the ruin probabilities (1.4), (1.5), and (1.6) for the multivariate compound Poisson risk model in which the claim sizes \( X_{1,n}, \ldots, X_{s,n} \) are correlated and have a multivariate phase type distribution (MPH) in the sense of Assaf et al. (1984). As in the univariate case, the multivariate phase type distributions also have some closure properties that induce a tractable explicit expression for (1.4). We also discuss how the dependence among various types of claims would affect the ruin probabilities, and obtain the bounds for these ruin probabilities using stochastic comparison methods. Our results show that ignoring dependence among various claims of a risk model in a common random environment often results in under-estimating the ruin probabilities.

A two-dimensional case of the multivariate compound Poisson risk model has been discussed by Chan et al. (2003), in which the claim sizes \( X_{1,n} \) and \( X_{2,n} \) are assumed to be independent. They obtained an explicit formula for \( \psi_{\text{sum}}(u) \) for the risk models with the claims of phase type, and discussed a simple lower bound for \( \psi_{\text{or}}(u_1, u_2) \). Our model is the multivariate extension of the model discussed in Chan et al. (2003), also allowing a general dependence structure among various claims, and our dependence analysis yields a simple upper bound for certain \( \psi_{\text{or}}(u_1, u_2) \). It is also worth mentioning that there are some studies in the literature on the dependence structure of multivariate risk processes (see Belzunce et al. (2002) and the references therein). In contrast, however, our analysis focuses on the effect of the claim dependence on the ruin probabilities in infinite horizon.

The paper is organized as follows. After a brief discussion on the multivariate phase type distributions, Section 2 gives an explicit representation for the convolution of a multivariate
phase type distribution, and hence presents a formula of $\psi_{\text{sum}}(u)$ for the risk models with the claims of multivariate phase type. Section 2 also discusses the effect of the claim dependence on $\psi_{\text{sum}}(u)$ by using the convex and supermodular comparison methods on the claim size vectors. Section 3 discusses the association property of the multivariate claim surplus process and presents the bounds for $\psi_{\text{or}}(u_1, \ldots, u_s)$ and $\psi_{\text{and}}(u_1, \ldots, u_s)$. Finally, Section 4 concludes the paper with a risk model with multivariate exponentially distributed claims and some illustrative examples. Throughout this paper, the term ‘increasing’ and ‘decreasing’ mean ‘nondecreasing’ and ‘nonincreasing’ respectively, and the measurability of sets and functions as well as the existence of expectations are often assumed without explicit mention.

2 Multivariate Risk Model of Phase Type

Let $X = \{X(t), t \geq 0\}$ be a right-continuous, continuous-time Markov chain on a finite state space $\mathcal{E}$ with generator $Q$. Let $\mathcal{E}_i$, $i = 1, \ldots, s$, be $s$ nonempty stochastically closed subsets of $\mathcal{E}$ such that $\cap_{i=1}^s \mathcal{E}_i$ is a proper subset of $\mathcal{E}$ (A subset of the state space is said to be stochastically closed if once the process $X$ enters it, $X$ never leaves). We assume that absorption into $\cap_{i=1}^s \mathcal{E}_i$ is certain. Since we are interested in the process only until it is absorbed into $\cap_{i=1}^s \mathcal{E}_i$, we may assume, without loss of generality, that $\cap_{i=1}^s \mathcal{E}_i$ consists of one state, which we shall denote by $\Delta$. Thus, without loss of generality, we may write $\mathcal{E} = \mathcal{E}_0 \cup (\cup_{i=1}^s \mathcal{E}_i)$ for some subset $\mathcal{E}_0 \subset \mathcal{E}$ with $\mathcal{E}_0 \cap \mathcal{E}_j = \emptyset$ for $1 \leq j \leq s$. The states in $\mathcal{E}$ are enumerated in such a way that the absorbing state $\Delta$ is the first element of $\mathcal{E}$. Thus, the generator of the chain has the form

$$Q = \begin{bmatrix} 0 & 0 \\ -A & A \end{bmatrix},$$

where $0$ is the row vector of zeros of $|\mathcal{E}| - 1$ dimension, and $e$ is the column vector of 1’s, and $A$ is an $((|\mathcal{E}| - 1) \times (|\mathcal{E}| - 1)$ matrix. Let $\gamma$ be an initial probability vector on $\mathcal{E}$ such that $\gamma(\Delta) = 0$. Then we can write $\gamma = (0, \alpha)$.

We define

$$X_i = \inf\{t \geq 0 : X(t) \in \mathcal{E}_i\}, \quad i = 1, \ldots, s. \quad (2.1)$$

As in Assaf et al. (1984), for simplicity, we shall assume that $P(X_1 > 0, \ldots, X_s > 0) = 1$. The joint distribution of $(X_1, \ldots, X_s)$ is called a multivariate phase type distribution with representation $(\alpha, A, \mathcal{E}, \mathcal{E}_i, i = 1, \ldots, s)$, and $(X_1, \ldots, X_s)$ is called a phase type random vector. The set of all multivariate distributions arising in this fashion is denoted by MPH.

When $s = 1$, the distribution of (2.1) reduces to the univariate phase type (PH) distribution introduced in Neuts (1981) (See Section 1). Examples of multivariate phase type
distributions include, among many others, the well-known Marshall-Olkin distribution (Marshall and Olkin 1967). The multivariate phase type distributions, their properties, and some related applications in reliability theory were discussed in Assaf et al. (1984). As in the univariate case, those multivariate phase type distributions (and their densities, Laplace transforms and moments) can be written in a closed form. The set of s-dimensional PH distributions is dense in the set of all distributions on \([0, \infty)^s\). It is also shown in Assaf et al. (1984) that MPH is closed under marginalization, finite mixtures, and the formation of coherent reliability systems.

The question of whether MPH is closed under finite convolutions is left unsettled in Assaf et al. (1984). Using the idea of accumulated rewards, Kulkarni (1989) introduced a class of multivariate phase type distributions that is strictly larger than MPH, and showed that this larger class is closed under linear transforms. Kulkarni’s result, in particular, established the following closure property: If \((X_1, \ldots, X_s)\) has a distribution in MPH, then \(\sum_{i=1}^s X_i\) has a phase type distribution. Kulkarni’s proof is constructive, but offers no direct representation of \(\sum_{i=1}^s X_i\) in terms of the distributional parameters of \((X_1, \ldots, X_s)\). Since this representation is crucial for us to obtain the explicit formula for the ruin probability \(\psi_{\text{sum}}(u)\) in our multivariate risk models of phase type, we now provide a direct proof for this closure property. For this, we partition the state space as follows.

\[
\Gamma_0^s = \mathcal{E}_0.
\]

\[
\Gamma_i^{s-1} = \mathcal{E}_i - \bigcup_{k \neq i} (\mathcal{E}_i \cap \mathcal{E}_k), i = 1, \ldots, s.
\]

\[
\Gamma_{ij}^{s-2} = \mathcal{E}_i \cap \mathcal{E}_j - \bigcup_{k \neq i, k \neq j} (\mathcal{E}_i \cap \mathcal{E}_j \cap \mathcal{E}_k), i \neq j.
\]

\[
\ldots, 
\]

For any \(S \subseteq \{1, \ldots, s\}\),

\[
\Gamma_S^{s-|S|} = \bigcap_{i \in S} \mathcal{E}_i - \bigcup_{k \notin S} (\bigcap_{i \in S} \mathcal{E}_i \cap \mathcal{E}_k).
\]

\[
\ldots, 
\]

\[
\Gamma_{12\ldots s}^0 = \{\Delta\}.
\]

In other words, \(\Gamma_S^{s-|S|}\) contains the states only in \(\mathcal{E}_i\) for all \(i \in S\), but not in any other \(\mathcal{E}_j, j \notin S\). Note that these \(\Gamma\)’s form a partition of \(\mathcal{E}\). For each state \(e \in \mathcal{E}\), define,

\[
k(e) = \text{number of indexes in } \{j : e \notin \mathcal{E}_j, 1 \leq j \leq s\}.
\]

(2.2)

For example, \(k(e) = s\) for all \(e \in \Gamma_0^s\), \(k(e) = 0\) for all \(e \in \Gamma_{12\ldots s}^0\), and in general, \(k(e) = s - |S|\) for all \(e \in \Gamma_S^{s-|S|}\).
Lemma 2.1 Let \((X_1, \ldots, X_n)\) be a phase type vector whose distribution has representation 
\((\alpha, A, \mathcal{E}, \mathcal{E}_i, i = 1, \ldots, s)\), where \(A = (a_{e,e'})\). Then \(\sum_{i=1}^s X_i\) has a phase type distribution with representation 
\((\alpha, T, |\mathcal{E}| - 1)\), where \(T = (t_{e,e'})\) is given by,

\[
t_{e,e'} = \frac{a_{e,e'}}{k(e)}, \tag{2.3}
\]

that is, \(t_{e,e'} = \frac{a_{e,e'}}{k(e)}\) if \(e \in \Gamma_S^k\), for some \(S\).

**Proof.** Let \(\{X(t), t \geq 0\}\) be an underlying Markov chain with state space \(\mathcal{E}\). The idea of our construction is based on the following two simple observations.

1. \(\sum_{i=1}^s X_i\) is the sum of \(k\) multiples of the occupation times spent by \(\{X(t), t \geq 0\}\) in \(\Gamma^k_S\) 
   over \(k = 1, \ldots, s, S \subseteq \{1, \ldots, s\}\), until it is absorbed into \(\Delta\).

2. A \(k\) multiple of exponentially distributed random variable with mean \(1/\lambda\) has an exponential distribution with mean \(k/\lambda\).

Let \(\{Z_n, n \geq 0\}\) be the embedded Markov chain of \(\{X(t), t \geq 0\}\). That is, \(Z_n\) is the state which \(\{X(t), t \geq 0\}\) visits immediately after its \(n\)th jump epoch. Let

\[
N = \min\{n : Z_n = \Delta\}, \\
N_i = \min\{n : Z_n \in \mathcal{E}_i\}, i = 1, \ldots, s.
\]

Thus, \(Z_0, Z_1, \ldots, Z_N\) are the states visited until final absorption in \(\Delta\). Let \(Y_n\) be the sojourn time of \(\{X(t), t \geq 0\}\) at \(Z_n\), \(0 \leq n \leq N - 1\). Then we have \(X_i = Y_0 + Y_1 + \ldots + Y_{N_i-1}\), for \(1 \leq i \leq s\), and thus

\[
\sum_{i=1}^s X_i = \sum_{i=1}^s \sum_{n=0}^{N_i-1} Y_n. \tag{2.4}
\]

If \(Z_n \in \mathcal{E}_j\), then \(Y_n\) does not appear in the representation for \(X_j\), and conversely, if \(Y_n\) does not appear in the representation for \(X_j\) then \(N_j \leq n\) and so \(Z_n \in \mathcal{E}_j\) since \(\mathcal{E}_j\) is stochastically closed. Therefore, each \(Y_n\) in the sum (2.4) occurs exactly \(k(Z_n)\) times, we have

\[
\sum_{i=1}^s X_i = \sum_{n=0}^{N-1} k(Z_n) Y_n. \tag{2.5}
\]

Let \(\{X^*(t), t \geq 0\}\) be a continuous-time Markov chain with state space \(\mathcal{E}\), initial probability vector \((0,\alpha)\) and generator \(T\) given by (2.3). We now argue that the time for
\(\{X^*(t), t \geq 0\}\) to be absorbed in \(\Delta\) has the same distribution as that of \(\sum_{i=1}^s X_i\). Let \(\tau_n\) be the \(n\)th jump epoch in \(\{X^*(t), t \geq 0\}\), with \(\tau_0 = 0\). Consider the discrete-time Markov chain \(\{Z_n^*, n \geq 0\}\) with state space \(\mathcal{E}\), where

\[
Z_n^* = X^*(\tau_n+), n \geq 0.
\]
The embedded Markov chain \( \{Z_n^*, n \geq 0\} \) has the following transition probability matrix
\[
\begin{bmatrix}
1 & 0 \\
p_0 & P
\end{bmatrix},
\]
where the matrix \( P = (p_{e,e'}) \), and the column vector \( p_0 = (p_e) \) are defined by,
\[
p_{e,e'} = \begin{cases} 
\frac{i_{e,e'}^{e'}}{i_{e,e}} & \text{if } e \neq e' \\
0 & \text{if } e = e'
\end{cases}
\]
\[
p_e = 1 - \sum_{e' \in \mathcal{E} - \{\Delta\}} p_{e,e'}.
\]
Because of (2.3), we have
\[
p_{e,e'} = \begin{cases} 
\frac{\alpha_{e,e'}}{\alpha_{e,e}} & \text{if } e \neq e' \\
0 & \text{if } e = e'
\end{cases}
\]
and so \( \{Z_n^*, n \geq 0\} \) is stochastically identical to the embedded Markov chain \( \{Z_n, n \geq 0\} \) of \( \{X(t), t \geq 0\} \).

Let \( M_e \) be the number of times that \( \{Z_n^*, n \geq 0\} \) visits state \( e \) until it is absorbed in \( \Delta \). Let \( \{Y_{e,n}^*, n \geq 1, e \in \mathcal{E} - \{\Delta\}\} \) be a collection of independent random variables, with \( Y_{e,n}^* \) exponentially distributed with mean \( k(e)/(-\alpha_{e,e}) \). The time \( T^* \) for \( \{X^*(t), t \geq 0\} \) to absorption in \( \Delta \) can be expressed as follows,
\[
T^* = st \sum_{e \in \mathcal{E} - \{\Delta\}} M_e \sum_{n=1}^{M_e} Y_{e,n}^*,
\]
here and in the sequel, “\( =_{st} \)” denotes the equality in distribution. Observe that \( Y_{e,n}^* =_{st} k(e)Y_{e,n} \), where \( \{Y_{e,n}, n \geq 1, e \in \mathcal{E} - \{\Delta\}\} \) is a collection of independent random variables, with \( Y_{e,n} \) exponentially distributed with mean \( 1/(-\alpha_{e,e}) \). It now follows from (2.5) that
\[
T^* =_{st} \sum_{e \in \mathcal{E} - \{\Delta\}} M_e \sum_{n=1}^{M_e} k(e)Y_{e,n}
=_{st} \sum_{n=0}^{N-1} k(Z_n)Y_n = \sum_{i=1}^{s} X_i.
\]
Therefore, \( \sum_{i=1}^{s} X_i \) is a phase type random variable with representation \( (\alpha, T, |\mathcal{E}| - 1) \).

Lemma 2.1 can be used to obtain the explicit expression of the ruin probability \( \psi_{\text{sum}}(u) \) for the multivariate compound Poisson risk model of phase type.

First, we point out that the ruin probability \( \psi_{\text{sum}}(u) \) can be reduced to the ruin probability in the compound Poisson risk model since
\[
\psi_{\text{sum}}(u) = P \left( \sup_{0 \leq t < \infty} \left\{ \sum_{j=1}^{s} S_j^i(t) \right\} > u \right)
= P \left( \sup_{0 \leq t < \infty} \left\{ \sum_{n=1}^{N(t)} (X_{1,n} + \cdots + X_{s,n}) - ct \right\} > u \right).
\]
where $c = \sum_{j=1}^{s} p_j$. Then, together with Lemma 2.1 and (1.3), we immediately obtain the explicit formula for $\psi_{\text{sum}}(u)$.

**Theorem 2.2** For the multivariate compound Poisson risk model with phase type claim vectors having representation $(\alpha, A, \mathcal{E}, \epsilon_i, i = 1, \ldots, s)$, the ruin probability $\psi_{\text{sum}}(u)$ is given by

$$
\psi_{\text{sum}}(u) = -\frac{\lambda}{c} T^{-1} e^{(T - t_0 \alpha T^{-1})u} e,
$$

where $T$ is given by (2.3), $t_0 = -Te$, and $c = \sum_{j=1}^{s} p_j$.

Theorem 2.2 provides a formula to calculate $\psi_{\text{sum}}(u)$, but it is not clear how the dependence among various claims would affect the ruin probability. To study this, we utilize the comparison methods. The following stochastic orders are most relevant to this research.

**Definition 2.3** Let $X = (X_1, \ldots, X_s)$ and $Y = (Y_1, \ldots, Y_s)$ be two $\mathcal{R}^s$-valued random vectors.

1. $X$ is said to be larger than $Y$ in stochastic order (convex order, increasing convex order), denoted by $X \geq_{st} (\geq_{cx}, \geq_{icx}) Y$, if $Ef(X) \geq Ef(Y)$ for all increasing (convex, increasing convex) functions $f$.

2. $X$ is said to be more dependent than $Y$ in supermodular order, denoted by $X \geq_{sm} Y$, if $Ef(X) \geq Ef(Y)$ for all supermodular functions $f$; that is, functions satisfying that for all $x, y \in \mathcal{R}^s$,

$$
f(x \vee y) + f(x \wedge y) \geq f(x) + f(y),
$$

where $x \vee y$ denotes the vector of component-wise maximums, and $x \wedge y$ denotes the vector of component-wise minimums.

These stochastic orders have many useful properties and applications, and are studied in details in Marshall and Olkin (1979), Shaked and Shanthikumar (1994), and Müller and Stoyan (2002), and references therein. The following properties are used in our next theorem and their proofs can be found in Marshall and Olkin (1979) and Shaked and Shanthikumar (1994).

**Lemma 2.4** Let $X = (X_1, \ldots, X_s)$ and $Y = (Y_1, \ldots, Y_s)$ be two $\mathcal{R}^s$-valued random vectors.

1. If $X \geq_{st} (\geq_{cx}) Y$, then $X \geq_{icx} Y$.

2. If $X \geq_{icx} Y$ and $E(X_j) = E(Y_j)$ for any $j = 1, \ldots, s$, then $X \geq_{cx} Y$. 

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3. If $X \geq_{sm} Y$, then $X_j$ and $Y_j$ have the same marginal distribution for any $j = 1, \ldots, s$, and

$$P(X_1 > x_1, \ldots, X_s > x_s) \geq P(Y_1 > x_1, \ldots, Y_s > x_s),$$

$$P(X_1 \leq x_1, \ldots, X_s \leq x_s) \geq P(Y_1 \leq x_1, \ldots, Y_s \leq x_s),$$

for any $(x_1, \ldots, x_s)$.

Thus, if $X \geq_{sm} Y$, then $Cov(X_i, X_j) \geq Cov(Y_i, Y_j)$ for any $i \neq j$, and $Corr(X_i, X_j) \geq Corr(Y_i, Y_j)$ for any $i \neq j$, where $Corr$ denotes the correlation coefficient.

Consider two multivariate compound Poisson risk models $M_1$ and $M_2$ introduced in Section 1. Suppose that $M_1$ and $M_2$ have the same claim event arrival rate, same premium rates, and same initial reserves, but different claim size vectors $X_n = (X_{1,n}, \ldots, X_{s,n})$ and $Y_n = (Y_{1,n}, \ldots, Y_{s,n})$ respectively. Let $\psi_{sum}^X(u)$ and $\psi_{sum}^Y(u)$ denote the ruin probabilities of type $\psi_{sum}(u)$ in $M_1$ and $M_2$ respectively. Note here that $X_n$ and $Y_n$ may have general distributions.

It is easy to see that if $X_n \geq_{st} Y_n$, then $\psi_{sum}^X(u) \geq \psi_{sum}^Y(u)$ for any $u$. Further, we have the following comparison results under dependence orderings.

**Theorem 2.5**

1. If $X_n \geq_{cx} Y_n$, then $\psi_{sum}^X(u) \geq \psi_{sum}^Y(u)$ for any $u$.

2. If $X_n \geq_{sm} Y_n$, then $\psi_{sum}^X(u) \geq \psi_{sum}^Y(u)$ for any $u$.

**Proof.** It follows from a well-known stop-loss ordering result (see for example, Asmussen et al. 1995) that if $\sum_{j=1}^s X_{j,n} \geq_{icx} \sum_{j=1}^s Y_{j,n}$, then $\psi_{sum}^X(u) \geq \psi_{sum}^Y(u)$ for any $u \geq 0$. Thus, Theorem 2.5 (1) follows from the fact that $f(\sum_{j=1}^s x_j)$ is convex in $(x_1, \ldots, x_s)$ for any convex function $f$, and Theorem 2.5 (2) follows from the fact that $f(\sum_{j=1}^s x_j)$ is supermodular in $(x_1, \ldots, x_s)$ for any convex function $f$ (Marshall and Olkin 1979, or Müller and Stoyan 2002).

Therefore, if the claim sizes are more dependent in the sense of $\geq_{cx}$ or $\geq_{sm}$, then the ruin probability $\psi_{sum}$ becomes larger. A numerical example given at the end of the paper illustrates this effect of dependence.

### 3 Bounds for the Ruin Probabilities

It is very difficult to obtain explicit formulas for the other two types of ruin probabilities $\psi_{or}(u_1, \ldots, u_s)$ and $\psi_{and}(u_1, \ldots, u_s)$ even if various claims are independent exponential random variables. They can not be reduced to the ruin probability in an one-dimensional compound Poisson risk model. The situations at ruin in the two cases are more complicated.
than those at ruin in an one-dimensional compound Poisson risk model. However, we can obtain bounds for \( \psi_{or}(u_1, \ldots, u_s) \) and \( \psi_{and}(u_1, \ldots, u_s) \) in terms of \( \psi_j(u_j), j = 1, \ldots, s \), which are ruin probabilities in subportfolios, namely, for \( 1 \leq j \leq s \),

\[
\psi_j(u_j) = P \left( \sup_{0 \leq t < \infty} S^j(t) > u_j \right) = P \left( \sup_{0 \leq t < \infty} \left\{ \sum_{n=1}^{N(t)} X_{j,n} - p_j t \right\} > u_j \right).
\]

To do so, we need to establish the association properties for multivariate risk models. With that, we can obtain some simple bounds for the ruin probabilities \( \psi_{or}(u_1, \ldots, u_s) \) and \( \psi_{and}(u_1, \ldots, u_s) \) in multivariate risk models with positively associated claims.

First, we review some notions of associations of stochastic processes and random vectors, which can be found, for example, in Tong (1980) and Lindqvist (1988). Let \( \mathcal{E} \) be a partially ordered Polish space (that is, a complete, separable metric space) with a closed partial ordering \( \leq \).

**Definition 3.1**

1. A probability measure \( P \) on \( \mathcal{E} \) is said to be associated if for all upper subsets \( U_1 \) and \( U_2 \) of \( \mathcal{E} \), \( P(U_1 \cap U_2) \geq P(U_1)P(U_2) \) (A subset \( U \subseteq \mathcal{E} \) is called upper if \( x \in U \) and \( x \leq y \) imply that \( y \in U \)).

2. An \( \mathcal{E} \)-valued random variable \( X \) is said to be associated if the probability measure \( P_X \) on \( \mathcal{E} \) induced by \( X \) is associated in the sense of Definition 3.1 (1).

Some properties of association are summarized below.

**Lemma 3.2** (Lindqvist 1988) Let \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) be partially ordered Polish spaces.

1. If a probability measure \( P \) on \( \mathcal{E}_1 \) is associated and \( f : \mathcal{E}_1 \to \mathcal{E}_2 \) is increasing, then the induced measure \( Pf^{-1} \) on \( \mathcal{E}_2 \) is associated.

2. If a probability measure \( P_1 \) on \( \mathcal{E}_1 \) is associated and a probability measure \( P_2 \) on \( \mathcal{E}_2 \) is associated, then the usual product measure \( P_1 \times P_2 \) on \( \mathcal{E}_1 \times \mathcal{E}_2 \) is associated.

3. Let \( X \) be an \( \mathcal{E}_1 \)-valued random variable and \( Y \) be an \( \mathcal{E}_2 \)-valued random variable. If \( X \) is associated, \( (Y \mid X = x) \) is associated for all \( x \) and \( E[f(Y) \mid X = x] \) is increasing in \( x \) for all increasing function \( f \), then \( Y \) is associated.

As indicated in Lindqvist (1988), Definition 3.1 (1) is equivalent to the corresponding statement with the upper set being replaced by the lower set (A subset \( U \subseteq \mathcal{E} \) is called lower if \( x \in U \) and \( x \geq y \) imply that \( y \in U \)). Thus if the mapping \( f \) in Lemma 3.2 (1) and (3) is replaced by a decreasing mapping, the results still hold.

Let \( X = \{X_n, n \geq 0\} \) be a discrete-time stochastic process where \( X_n \) is \( \mathcal{E} \)-valued for all \( n \geq 0 \). Let \( \mathcal{E}^\infty = \mathcal{E} \times \mathcal{E} \times \ldots \) be the product space of infinitely many \( \mathcal{E} \)'s with the usual product
topology and the coordinate-wise partial ordering; that is, for any \( x = (x_1, x_2, \ldots) \in \mathcal{E}^\infty \)
and \( y = (y_1, y_2, \ldots) \in \mathcal{E}^\infty \), \( x \leq y \) if and only if \( x_i \leq y_i \) for all \( i \geq 1 \). The product space \( \mathcal{E}^\infty \) is again a partially ordered Polish space (see Billingsley, 1968, Page 218). A process \( X \)
is said to be associated if the probability measure \( P_X \) on \( \mathcal{E}^\infty \) induced by \( X \) is associated in
the sense of Definition 3.1 (1).

For a continuous-time \( \mathcal{E} \)-valued process \( X = \{X(t), 0 \leq t \leq M\} \) where \( M \) is a positive
real number, we need to consider the space \( D_{\mathcal{E}}[0,M] \), the space of all functions from the real
interval \([0,M]\) to \( \mathcal{E} \) which are right continuous and have left limits. The space \( D_{\mathcal{E}}[0,M] \) is
a partially ordered Polish space with the Skorohod metric and the partial order \( \leq \) defined
as \( x \leq y \) if \( x(t) \leq y(t) \) for all \( t \in [0,M] \) where \( x, y \in D_{\mathcal{E}}[0,M] \). Note that the \( D_{\mathcal{E}}[0,\infty) \)
is still Polish, where we use the Stone (1963) modification of Skorohod’s metric (Kamae et al.
(1977)). A process \( X = \{X(t), 0 \leq t \leq M\} \) is called a process with \( D_{\mathcal{E}}[0,M] \)-sample paths
if all the sample paths of \( X \) are right continuous and have left limits. A continuous-time
process \( X = \{X(t), 0 \leq t \leq M\} \) with \( D_{\mathcal{E}}[0,M] \)-sample paths is said to be associated if the
probability measure \( P_X \) on \( D_{\mathcal{E}}[0,M] \) induced by \( X \) is associated in the sense of Definition
3.1 (1).

In order to verify the association property of a stochastic process, we often need the
following:

**Definition 3.3** Let \( \mathcal{E} \) be a partially ordered Polish space.

1. The \( \mathcal{E} \)-valued random variables \( Z_1, \ldots, Z_k \) are said to be associated if the probability
measure \( P \) on the space \( \mathcal{E}^k \) induced by the random vector \( (Z_1, \ldots, Z_k) \) is associated in
the sense of Definition 3.1 (1).

2. (Esary and Proschan 1970) An \( \mathcal{E} \)-valued stochastic process \( \{Z(t), t \in \mathcal{I}\} \) is said to
be associated in time if for any set \( \{t_1, \ldots, t_k\} \subseteq \mathcal{I} \subseteq [0, \infty) \), the random variables
\( Z(t_1), \ldots, Z(t_k) \) are associated in the sense of (1).

It was shown in Lindqvist (1988) that a probability measure \( P \) is associated on a parti-
cially ordered Polish space \( \mathcal{E} \) if and only if \( \int fg dP \leq (\int f dP)(\int g dP) \) for all real bounded
increasing functions \( f \) and \( g \) defined on \( \mathcal{E} \). Therefore, the probability measure \( P \) induced
by an \( \mathcal{R}^n \)-valued random vector \( Z = (Z_1, \ldots, Z_n) \) is associated if and only if the covariance
\( \text{Cov}(f(Z), g(Z)) \geq 0 \), for all increasing functions \( f \) and \( g \) (Esary et al. (1967)). Note also
that the probability measure \( P_Z \) induced by a process \( Z = \{Z(t), t \in \mathcal{I}\} \) on an appropriate
state space is associated if and only if

\[
E(f(Z)g(Z)) \geq Ef(Z)Eg(Z)
\]  

(3.1)
for all bounded increasing functions $f$ and $g$. This implies that the process $Z$ is associated in time. Conversely, Lindqvist (1987, 1988) has showed that if $Z$ is associated in time, then (3.1) holds under certain conditions. Specifically, we have,

**Lemma 3.4** (Lindqvist 1988)

1. A discrete-time process $X = \{X_n, n \geq 0\}$ with state space $\mathcal{R}^s$ is associated if and only if $X$ is associated in time.

2. A process $X = \{X(t), 0 \leq t \leq M\}$ with $D_{\mathcal{R}}[0, M]$-sample paths is associated if and only if $X$ is associated in time.

We are now in position to establish our main result in this section.

**Theorem 3.5** For the multivariate compound Poisson risk model with associated claim vectors, we have

$$\min_{1 \leq j \leq s} \{\psi_j(u_j)\} \geq \psi_{\text{and}}(u_1, \ldots, u_s) \geq \prod_{j=1}^{s} \psi_j(u_j)$$

for any non-negative $u_1, \ldots, u_s$.

**Proof.** The upper bound is obvious and holds for general claim vectors, and we only need to establish the lower bound. Let $S^j(t) = \sum_{n=1}^{N(t)} X_{j,n} - p_j t$ be the type $j$ claim surplus at time $t$, $1 \leq j \leq s$, where $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda$. We first show that the multivariate surplus process $\{(S^1(t), \ldots, S^s(t)), t \geq 0\}$ is associated.

Since $N(t) = \max\{n : \sum_{i=1}^{n} E_i \leq t\}$ for any $t$, where $E_i, i = 1, 2, \ldots$, are independent and identically distributed with exponential distributions, then $\{N(t), t \geq 0\}$ is a decreasing transform of $\{E_i, i \geq 1\}$. From Lemma 3.2 (2), we know that $\{E_i, i \geq 1\}$ is associated in time, and hence, by Lemma 3.4, $\{E_i, i \geq 1\}$ is associated. It then follows from Lemma 3.2 (1) that $\{N(t), t \geq 0\}$ is associated.

Let $S = \{(S^1(t), \ldots, S^s(t)), t \geq 0\}$. Consider

$$[S \mid N(t) = n_t, t \geq 0] = \left[ \sum_{n=1}^{n_t} X_{1,n} - p_1 t, \ldots, \sum_{n=1}^{n_t} X_{s,n} - p_s t, t \geq 0 \right],$$

for any $\{n_t, t \geq 0\} \in D_{\mathcal{R}}[0, \infty)$. If $n_t \leq n'_t$ for any $t \geq 0$, then we have

$$\left[ \sum_{n=1}^{n_t} X_{1,n} - p_1 t, \ldots, \sum_{n=1}^{n_t} X_{s,n} - p_s t \right] \leq \left[ \sum_{n=1}^{n'_t} X_{1,n} - p_1 t, \ldots, \sum_{n=1}^{n'_t} X_{s,n} - p_s t \right], t \geq 0,$$

almost surely. Thus $E[f(S) \mid N(t) = n_t, t \geq 0]$ is increasing in $\{n_t, t \geq 0\} \in D_{\mathcal{R}}[0, \infty)$, for all increasing functionals $f$. 

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Because \((X_{1,n}, \ldots, X_{s,n})\) is associated for any given \(n\), it follows from Lemmas 3.2 (2) and 3.4 that \(\{(X_{1,n}, \ldots, X_{s,n}), n \geq 1\}\) is associated. Then, by Lemma 3.2 (1), we have that \([ S \mid N(t) = n_t, t \geq 0]\) is associated for fixed \(\{n_t, t \geq 0\} \in D_K[0, \infty)\).

From Lemma 3.2 (3), \(S = \{(S^1(t), \ldots, S^s(t)), t \geq 0\}\) is associated. Since

\[
\left(\sup_{0 \leq t < \infty} S^1(t), \ldots, \sup_{0 \leq t < \infty} S^s(t)\right)
\]

is an increasing transform of \(\{(S^1(t), \ldots, S^s(t)), t \geq 0\}\), then this random vector is associated, which implies that

\[
P\left(\sup_{0 \leq t < \infty} S^1(t) > u_1, \ldots, \sup_{0 \leq t < \infty} S^s(t) > u_s\right) \geq \prod_{j=1}^{s} P\left(\sup_{0 \leq t < \infty} S^j(t) > u_j\right),
\]

that is, \(\psi_{\text{and}}(u_1, \ldots, u_s) \geq \prod_{j=1}^{s} \psi(u_j)\) for any non-negative \(u_1, \ldots, u_s\). ■

Li (2003) obtained the following sufficient condition for a phase type random vector to be associated.

**Lemma 3.6** Let \((X_1, \ldots, X_s)\) be the random vector defined by (2.1) where the state space \(E\) of the underlying Markov chain \(X = \{X(t), t \geq 0\}\) is partially ordered with the consistent order \(\leq\). If \(X\) is stochastically monotone, and up-down with respect to \(\leq\), and starts at state \(x\) almost surely, for some \(x \in E_0\), then \((X_1, \ldots, X_s)\) is associated.

The consistent order is a natural extension of the coordinate-wise order to a general partially ordered set, and its detailed description can be found in Li (2003). Thus, for the compound Poisson model with phase type claim vectors satisfying the condition in Lemma 3.6, we can use the explicit, univariate ruin probabilities to bound the multivariate ruin probability \(\psi_{\text{and}}(u_1, \ldots, u_s)\).

Using Theorem 3.5, we can also obtain an upper bound for \(\psi_{\text{or}}(u_1, \ldots, u_s)\). To illustrate this, consider the bivariate case,

\[
\psi_{\text{or}}(u_1, u_2) = \psi_1(u_1) + \psi_2(u_2) - \psi_{\text{and}}(u_1, u_2).
\]

If the claim vector is associated, then \(\psi_{\text{and}}(u_1, u_2) \geq \psi_1(u_1)\psi_2(u_2)\), and hence,

\[
\psi_{\text{or}}(u_1, u_2) \leq \psi_1(u_1) + \psi_2(u_2) - \psi_1(u_1)\psi_2(u_2).
\]

Thus, for the bivariate compound Poisson risk model with associated claim vectors, we have

\[
\max\{\psi_1(u_1), \psi_2(u_2)\} \leq \psi_{\text{or}}(u_1, u_2) \leq \psi_1(u_1) + \psi_2(u_2) - \psi_1(u_1)\psi_2(u_2).
\]

Both lower and upper bounds can be calculated explicitly for the bivariate phase type claims.
4 Multivariate Risk Model of Marshall-Olkin Type

In this section, we discuss an example of multivariate phase type risk models, where the claim size vectors have multivariate Marshall-Olkin exponential distributions. The formula for \( \psi_{\text{sum}}(u) \), and bounds for \( \psi_{\text{and}}(u_1, \ldots, u_s) \) are explicitly presented.

Let \( \{E_S, S \subseteq \{1, \ldots, s\}\} \) be a sequence of independent, exponentially distributed random variables, with \( E_S \) having mean \( 1/\lambda_S \). Let

\[
X_j = \min\{E_S : S \ni j\}, \quad j = 1, \ldots, s.
\]  

(4.1)

The joint distribution of \( (X_1, \ldots, X_s) \) is called the Marshall-Olkin exponential distribution with parameters \( \{\lambda_S, S \subseteq \{1, \ldots, s\}\} \) (Marshall and Olkin 1967). In the reliability context, \( X_1, \ldots, X_s \) can be viewed as the lifetimes of \( s \) components operating in a random shock environment where a fatal shock governed by Poisson process \( \{N_S(t), t \geq 0\} \) with rate \( \lambda_S \) destroys all the components with indexes in \( S \subseteq \{1, \ldots, s\} \) simultaneously. Assume that these Poisson shock arrival processes are independent, then,

\[
X_j = \inf\{t : N_S(t) \geq 1, S \ni j\}, \quad j = 1, \ldots, s.
\]  

(4.2)

Let \( \{M_S(t), t \geq 0\}, S \subseteq \{1, \ldots, s\} \), be independent Markov chains with absorbing state \( \Delta_S \), each representing the exponential distribution with parameter \( \lambda_S \). It follows from (4.2) that \( (X_1, \ldots, X_s) \) is of phase type with the underlying Markov chain on the product space of these independent Markov chains with absorbing classes \( E_j = \{(e_S) : e_S = \Delta_S \text{ for some } S \ni j\} \), \( 1 \leq j \leq s \).

It is easy to verify that the Marshall-Olkin exponentially distributed random vector is associated and the marginal distribution of its \( j \)th component is exponential with mean \( 1/\sum_{S:S\ni j} \lambda_S \).

It follows from Theorem 3.5 and (1.2) that for the multivariate compound Poisson risk model with Marshall-Olkin claim vectors, we have

\[
\psi_{\text{and}}(u_1, \ldots, u_s) \geq \left( \prod_{j=1}^{s} \frac{1}{1 + \theta_j} \right) \exp \left( -\sum_{j=1}^{s} \left( \frac{\theta_j}{1 + \theta_j} \left( \sum_{S:S\ni j} \lambda_S \right) u_j \right) \right)
\]

and

\[
\psi_{\text{and}}(u_1, \ldots, u_s) \leq \min_{1 \leq j \leq s} \left\{ \frac{1}{1 + \theta_j} \exp \left( -\frac{\theta_j}{1 + \theta_j} \left( \sum_{S:S\ni j} \lambda_S \right) u_j \right) \right\}
\]

for any non-negative \( u_1, \ldots, u_s \), where the relative security loading \( \theta_j = \left( \sum_{S:S\ni j} \lambda_S \right) p_{ij}/\lambda - 1, \) \( 1 \leq j \leq s \).
For a symmetric compound Poisson risk model with Marshall-Olkin claim size vectors, where \( \theta_j = \theta \) and \( \sum_{S:3\leq j} \lambda_S = \lambda' \) for all \( 1 \leq j \leq s \), we have

\[
\frac{1}{1 + \theta} \exp \left( -\frac{\theta \lambda'}{1 + \theta} \max_{1 \leq j \leq s} \{ u_j \} \right) \geq \psi_{\text{and}}(u_1, \ldots, u_s) \geq \left( \frac{1}{1 + \theta} \right)^s \exp \left( -\frac{\theta \lambda'}{1 + \theta} \sum_{j=1}^{s} u_j \right)
\]

for any non-negative \( u_1, \ldots, u_s \). Note that when \( \theta \) is very small, the upper and lower bounds are very close. Indeed,

\[
\lim_{\theta \to 0} \frac{1}{1 + \theta} \exp \left( -\frac{\theta \lambda'}{1 + \theta} \max_{1 \leq j \leq s} \{ u_j \} \right) = 1.
\]

To calculate \( \psi_{\text{sum}}(u) \), we need to simplify the underlying Markov chain for the Marshall-Olkin distribution and obtain its phase type representation. Let \( \{ X(t), t \geq 0 \} \) be a Markov chain with state space \( \mathcal{E} = \{ S : S \subseteq \{1, \ldots, s\} \} \), and starting at \( \emptyset \) almost surely. The index set \( \{1, \ldots, s\} \) is the absorbing state, and

\[
\mathcal{E}_0 = \{ \emptyset \}
\]

\[
\mathcal{E}_j = \{ S : S \ni j \}, \quad j = 1, \ldots, s.
\]

It follows from (4.2) that its sub-generator is given by \( A = (a_\epsilon, \epsilon') \), where

\[
a_\epsilon, \epsilon' = \sum_{L : L \subseteq S', L \cup S = S'} \lambda_L, \quad \text{if } \epsilon = S, \epsilon' = S' \text{ and } S \subset S',
\]

\[
a_\epsilon, \epsilon = \sum_{L : L \subseteq S} \lambda_L - \Lambda, \quad \text{if } \epsilon = S \text{ and } \Lambda = \sum_S \lambda_S,
\]

and zero otherwise. Using Theorem 2.2 and these parameters, \( \psi_{\text{sum}}(u) \) can be calculated. To illustrate the result, we consider the bivariate case.

When \( s = 2 \), the state space \( \mathcal{E} = \{12, 2, 1, 0\} \) and \( \mathcal{E}_j = \{12, j\}, j = 1, 2 \), where 12 is the absorbing state. The initial probability vector is \((0, 0, 0, 1)\), and its sub-generator is given by

\[
A = \begin{bmatrix}
-\lambda_{12} - \lambda_1 & 0 & 0 \\
0 & -\lambda_{12} - \lambda_2 & 0 \\
\lambda_2 & \lambda_1 & -\Lambda + \lambda_0
\end{bmatrix},
\]

where \( \Lambda = \lambda_{12} + \lambda_2 + \lambda_1 + \lambda_0 \). Thus, the matrix \( T \) in Theorem 2.2 is given by

\[
T = \begin{bmatrix}
-\lambda_1 - \lambda_{12} & 0 & 0 \\
0 & -\lambda_2 - \lambda_{12} & 0 \\
\frac{\lambda_2}{2} & \frac{\lambda_2}{2} & -\frac{\Delta'}{2}
\end{bmatrix},
\]
where $\Lambda_0 = \lambda_1 + \lambda_2 + \lambda_{12} = \Lambda - \lambda_0$. Hence,

$$T^{-1} = \begin{bmatrix}
-\frac{1}{\lambda_1 + \lambda_{12}} & 0 & 0 \\
0 & -\frac{1}{\lambda_2 + \lambda_{12}} & 0 \\
-\frac{\lambda_2}{(\lambda_1 + \lambda_{12})\lambda_0} & -\frac{\lambda_1}{(\lambda_2 + \lambda_{12})\lambda_0} & -\frac{2}{\lambda_0}
\end{bmatrix}.$$

Assume that $\lambda = 2$ and $p_1 = p_2 = 1$, and so $\frac{\lambda}{c} = 1$. From Theorem 2.2 with $\alpha = (0,0,1)$ and $B = T + \frac{\lambda}{c}(Te)(\alpha T^{-1})$, and after some algebra, we obtain

$$B = \begin{bmatrix}
-\lambda_1 - \lambda_{12} + \frac{\lambda_2}{\lambda_0} & \frac{\lambda_1(\lambda_2 + \lambda_{12})}{(\lambda_2 + \lambda_{12})\lambda_0} & \frac{2(\lambda_1 + \lambda_2)}{\lambda_0}
\frac{\lambda_2}{(\lambda_1 + \lambda_{12})\lambda_0} & -\lambda_2 - \lambda_{12} + \frac{\lambda_1}{\lambda_0} & \frac{2(\lambda_2 + \lambda_{12})}{\lambda_0}
\frac{\lambda_1}{(\lambda_1 + \lambda_{12})\lambda_0} & \frac{\lambda_1\lambda_{12}}{(\lambda_2 + \lambda_{12})\lambda_0} & \frac{\lambda_{12}}{\lambda_0} - \frac{\lambda_0}{2}
\end{bmatrix}$$

and

$$\psi_{\text{sum}}(u) = \left(-\frac{\lambda_2}{(\lambda_1 + \lambda_{12})\lambda_0}, -\frac{\lambda_1}{(\lambda_2 + \lambda_{12})\lambda_0}, -\frac{2}{\lambda_0}\right) e^{Bu} e. \quad (4.3)$$

To study the effect of dependence on $\psi_{\text{sum}}(u)$, consider two bivariate compound Poisson risk models $\mathcal{M}$ and $\mathcal{M}'$ of Marshall-Olkin type. Suppose that $\mathcal{M}$ and $\mathcal{M}'$ have the same claim event arrival rate, same premium rates, and same initial reserves, but different Marshall-Olkin claim size vectors with parameter sets $\{\lambda_{12}, \lambda_2, \lambda_1, \lambda_0\}$, and $\{\lambda'_{12}, \lambda'_2, \lambda'_1, \lambda'_0\}$ respectively. Let $\psi_{\text{sum}}(u)$ and $\psi'_{\text{sum}}(u)$ denote the ruin probabilities for total claim surplus in $\mathcal{M}$ and $\mathcal{M}'$ respectively.

Suppose that

$$\lambda'_{12} = \lambda_{12} - \delta \geq 0, \quad \lambda'_0 = \lambda_0 - \delta \geq 0, \quad \lambda'_1 = \lambda_1 + \delta, \quad \lambda'_2 = \lambda_2 + \delta. \quad (4.4)$$

It follows from Proposition 5.5 in Li and Xu (2000) that the claim size vector in $\mathcal{M}$ is more dependent than the claim size vector in $\mathcal{M}'$ in supermodular order. Thus, by Theorem 2.5, we have $\psi_{\text{sum}}(u) \geq \psi'_{\text{sum}}(u)$ for any $u \geq 0$. In particular, let $\delta = \lambda_{12} = \lambda_0$, we obtain that while keeping the marginal distributions of the claims of various types fixed, the ruin probability $\psi_{\text{sum}}(u)$ in a bivariate compound Poisson risk model with positively correlated, Marshall-Olkin exponentially distributed claim sizes is larger than that in a similar risk model with independent, exponentially distributed claim sizes. The latter was the focus in Chan et al. (2003). Hence, ignoring dependence among the claims of various types will often underestimate the ruin probabilities.

In Table 1, we calculate $\psi_{\text{sum}}(u)$ under several different sets of model parameters. Note that $\psi_{\text{sum}}(u)$ in (4.3) does not depend on $\lambda_0$. We introduce $\lambda_0$ in the model because we want to change the model parameters in a systematic fashion (4.4) according to supermodular
Table 1: Effects of Dependence on Ruin Probability

<table>
<thead>
<tr>
<th></th>
<th>(\psi_{\text{sum}}(u))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(u)</td>
</tr>
<tr>
<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>0.415080</td>
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<tr>
<td>3</td>
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<td>4</td>
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<td>5</td>
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<td>6</td>
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<td>7</td>
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</tr>
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<td>9</td>
<td>0.037966</td>
</tr>
<tr>
<td>10</td>
<td>0.026978</td>
</tr>
</tbody>
</table>

order, so that the effect of claim dependence on the ruin probability can be investigated. The analytic forms of \(\psi_{\text{sum}}(u)\) in the following three cases and the numerical values in the table were easily produced by Mathematica. The first column lists several initial reserves, and the second row lists the following three cases.

1. Case 1: \(\lambda_{12} = 0, \lambda_1 = \lambda_2 = 2.5, \lambda_0 = 0\). In this case, the claim vector \((X_{1,n}, X_{2,n})\) are independent, and

\[
\psi_{\text{sum}}(u) = -0.0221159 e^{-3.6583124 u} + 0.8221159 e^{-0.3416876 u}.
\]

2. Case 2: \(\lambda_{12} = 1, \lambda_1 = \lambda_2 = 1.5, \lambda_0 = 1\). In this case, the claim vector \((X_{1,n}, X_{2,n})\) are positively dependent, and

\[
\psi_{\text{sum}}(u) = -0.017786 e^{-3.1861407 u} + 0.8177864 e^{-0.3138593 u}.
\]

3. Case 3: \(\lambda_{12} = 2.5, \lambda_1 = \lambda_2 = 0, \lambda_0 = 2.5\). This is the comonotone case where \(X_{1,n} = X_{2,n}\), and so the claim vector \((X_{1,n}, X_{2,n})\) has the strongest positive dependence. In this case,

\[
\psi_{\text{sum}}(u) = 0.8 e^{-0.25 u}.
\]

In all the three cases, \((X_{1,n}, X_{2,n})\) has the same marginal distributions, namely, \(X_{1,n}\) and \(X_{2,n}\) have the exponential distributions with means \(1/(\lambda_{12} + \lambda_1)\) and \(1/(\lambda_{12} + \lambda_2)\), respectively, for all \(n = 1, 2, \ldots\). The only difference among them is the different correlation between \(X_{1,n}\)
and $X_{2n}$. It can be easily verified directly that the correlation coefficient of the claim size vector in Case 1 is smaller than that in Case 2, which, in turn, is smaller than that in Case 3. In fact, it follows from (4.4) that the claim size vector in Case 1 is less dependent than that in Case 2, which, in turn, is less dependent than that in Case 3, all in supermodular order. It is evident that the ruin probability $\psi_{\text{sum}}(u)$ becomes larger as the correlation grows.

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References


