An Exact Derivation of Jeans’ Criterion with Rotation for Gravitational Instabilities

Kohl Gill\textsuperscript{1} \quad David J. Wollkind\textsuperscript{1} \quad Bonni J. Dichone\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, Washington State University, Pullman, WA 99164-3113, USA

\textsuperscript{2}Department of Mathematics, Gonzaga University, 502 E. Boone Avenue MSC 2615, Spokane, WA 99258, USA
Abstract

An inviscid fluid model of a self-gravitating infinite expanse of a uniformly rotating adiabatic gas cloud consisting of the continuity, Euler’s, and Poisson’s equations for that situation is considered. There exists a static homogeneous density solution to this model relating that equilibrium density to the uniform rotation. A systematic linear stability analysis of this exact solution then yields a gravitational instability criterion equivalent to that developed by Sir James Jeans in the absence of rotation instead of the slightly more complicated stability behavior deduced by Subrahmanyan Chandrasekhar for this model with rotation, both of which suffered from the same deficiency in that neither of them actually examined whether their perturbation analysis was of an exact solution.
For the former case, it was not and, for the latter, the equilibrium density and uniform rotation were erroneously assumed to be independent instead of related to each other. Then this gravitational instability criterion is employed in the form of Jeans’ length to show that there is very good agreement between this theoretical prediction and the actual mean distance of separation of stars formed in the outer arms of the spiral galaxy Andromeda M31. Further the uniform rotation determined from the exact solution relation to equilibrium density and the corresponding rotational velocity for a reference radial distance are consistent with the spectroscopic measurements of Andromeda and the observational data of the spiral Milky Way galaxy.
A Self-Gravitational Uniformly-Rotating Adiabatic Inviscid Fluid Model of Infinite Extent

The governing equations for this situation are given by [2]:

**Continuity Equation:** \[ \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0; \]

**Euler’s Equation:** \[ \frac{D\mathbf{v}}{Dt} + 2\Omega \times \mathbf{v} + \Omega \times (\Omega \times \mathbf{r}) = -\frac{1}{\rho} \mathcal{P}'(\rho) \nabla \rho + \mathbf{g}; \]

**Poisson’s Equation:** \[ \nabla \cdot \mathbf{g} = -4\pi G_0 \rho. \]

The continuity and Euler’s equations follow from the conservation of mass and momentum for an inviscid fluid with the addition of the extra second and third terms on the left-hand side of the latter which represent the Coriolis effect and centrifugal force, respectively, due to the rotation [3]. Poisson’s equation follows from the divergence theorem and Newton’s law of universal gravitation [1].
Here $t \equiv$ time, $\mathbf{r} = (x, y, z) \equiv$ position vector, $\mathbf{\Omega} = (0, 0, \Omega_0) \equiv$ uniform rotation vector, $\rho \equiv$ density (mass/[unit volume]), $\mathbf{v} = (u, v, w) \equiv$ velocity vector with respect to the rotating frame, $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \equiv$ gradient operator, $D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla \equiv$ material derivative, $\mathcal{P}(\rho) = p_0 (\rho/\rho_0)^{\gamma_0} \equiv$ adiabatic pressure, $\mathbf{g} = -\nabla \varphi \equiv$ gravitational acceleration vector with $\varphi \equiv$ self-gravitating potential, and $G_0 \equiv$ universal gravitational constant.
The Governing Euler’s and Poisson’s Equations

Since
\[ \Omega \times v = \Omega_0(-v, u, 0), \]
\[ \Omega \times (\Omega \times r) = (\Omega \cdot r)\Omega - (\Omega \cdot \Omega)r = -\Omega_0^2(x, y, 0), \]
\[ \nabla \cdot g = -\nabla^2 \varphi; \]

the Euler’s and Poisson’s equations become:

Euler’s Equation: \[ \frac{Dv}{Dt} + 2\Omega_0(-v, u, 0) - \Omega_0^2(x, y, 0) \]
\[ = -\frac{1}{\rho} P'(\rho) \nabla \rho - \nabla \varphi; \]

Poisson’s Equation: \[ \nabla^2 \varphi = 4\pi G_0 \rho \] where \[ \nabla^2 = \nabla \cdot \nabla. \]
The Exact Static Homogeneous Density Solution

There exists an exact static homogeneous density solution of these equations of the form

\[ \mathbf{v} \equiv \mathbf{0} = (0, 0, 0), \quad \rho \equiv \rho_0, \varphi = \varphi_0 \]

where \( \varphi_0 \) satisfies

\[ \nabla \varphi_0 = \Omega_0^2 (x, y, 0), \quad \nabla^2 \varphi_0 = 4\pi G_0 \rho_0, \]

or

\[ \varphi_0(x, y) = \Omega_0^2 \frac{x^2 + y^2}{2} \quad \text{with} \quad \Omega_0^2 = 2\pi G_0 \rho_0 > 0. \]

This equilibrium state physically corresponds to the situation of a homogeneous quiescent gas for which the centrifugal and gravitational forces balance out each other.
Linear Perturbation Analysis of that Exact Solution

Now seeking a linear perturbation solution of these basic equations of the form

\[ \mathbf{v} = \varepsilon \mathbf{v}_1 + O(\varepsilon^2), \quad \rho = \rho_0[1 + \varepsilon s + O(\varepsilon^2)], \]

\[ \varphi = \varphi_0 + \varepsilon \varphi_1 + O(\varepsilon^2) \text{ where } \mathbf{v}_1 = (u_1, v_1, w_1); \]
...with $|\varepsilon| << 1$, we deduce that the perturbation quantities to this exact solution satisfy

$$
\frac{\partial s}{\partial t} + \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0;
$$

$$
\frac{\partial u_1}{\partial t} - 2\Omega_0 v_1 + c_0^2 \frac{\partial s}{\partial x} + \frac{\partial \varphi_1}{\partial x} = 0 \text{ where } c_0^2 = \mathcal{P}'(\rho_0) = \gamma_0 \frac{p_0}{\rho_0} > 0;
$$

$$
\frac{\partial v_1}{\partial t} + 2\Omega_0 u_1 + c_0^2 \frac{\partial s}{\partial y} + \frac{\partial \varphi_1}{\partial y} = 0;
$$

$$
\frac{\partial w_1}{\partial t} + c_0^2 \frac{\partial s}{\partial z} + \frac{\partial \varphi_1}{\partial z} = 0;
$$

$$
2\Omega_0^2 s - \nabla^2 \varphi_1 = 0.
$$
Normal Mode Solution of these Perturbation Equations

Assuming a normal mode solution for these perturbation quantities of the form

\[ [u_1, v_1, w_1, s, \varphi_1](x, y, z, t) = [A, B, C, E, F]e^{i(k_1x + k_2y + k_3z) + \sigma t}; \]

where \(|A|^2 + |B|^2 + |C|^2 + |E|^2 + |F|^2 \neq 0\), \(i = \sqrt{-1}\), and \(k_{1,2,3} \in \mathbb{R}\) to satisfy the implicit far-field boundedness property for those quantities; we obtain the following equations for \([A, B, C, E, F]\):

\[ ik_1A + ik_2B + ik_3C + \sigma E = 0; \]
\[ \sigma A - 2\Omega_0 B + ic_0^2k_1E + ik_1F = 0; \]
\[ 2\Omega_0 A + \sigma B + ic_0^2k_2E + ik_2F = 0; \]
\[ \sigma C + ic_0^2k_3E + ik_3F = 0; \]
\[ 2\Omega_0^2 E + k^2 F = 0 \text{ where } k^2 = k_1^2 + k_2^2 + k_3^2. \]
The Secular Equation or Dispersion Relation

Setting the determinant of the $5 \times 5$ coefficient matrix for the system of constants equal to zero to satisfy their nontriviality property, we obtain the secular equation

$$k^2 [\sigma^4 + (c_0^2 k^2 + 2\Omega_0^2)\sigma^2] + 4\Omega_0^2 (c_0^2 k^2 - 2\Omega_0^2) k_3^2 = 0.$$

Defining the wavenumber vector $\mathbf{k} = (k_1, k_2, k_3)$, its dot product with $\Omega$ satisfies

$$\mathbf{k} \cdot \Omega = k_3 \Omega_0 = |\mathbf{k}| |\Omega| \cos(\theta) = k \Omega_0 \cos(\theta),$$

$\theta$ being the azimuthal angle between $\mathbf{k}$ and $\Omega$; which implies that

$$k_3 = k \cos(\theta).$$

Then, substitution of this result and cancellation of $k^2$, yields

$$\sigma^4 + (c_0^2 k^2 + 2\Omega_0^2)\sigma^2 + 4\Omega_0^2 (c_0^2 k^2 - 2\Omega_0^2) \cos^2(\theta) = 0.$$
Stability Analysis of the Secular Equation

Since this secular equation is a quadratic in $\sigma^2$, we can first conclude that $\sigma^2 \in \mathbb{R}$ by showing that its discriminant

$$D = (c_0^2 k^2 + 2\Omega_0^2)^2 - 16\Omega_0^2 (c_0^2 k^2 - 2\Omega_0^2) \cos^2(\theta) \geq 0.$$ 

Consider the two cases of $c_0^2 k^2 - 2\Omega_0^2 \leq 0$ and $c_0^2 k^2 - 2\Omega_0^2 > 0$ separately. For the former case it is obvious, while for the latter one it can be deduced by noting that

$$D \geq (c_0^2 k^2 + 2\Omega_0^2)^2 - 16\Omega_0^2 (c_0^2 k^2 - 2\Omega_0^2) = (c_0^2 k^2 - 6\Omega_0^2)^2.$$
For $\theta = \pi/2$, we can conclude that $\sigma^2 = 0$ or $\sigma^2 = -(c_0^2 k^2 + 2\Omega_0^2) < 0$, while for $\theta \neq \pi/2$, the stability criteria governing such quadratics: Namely, given

$$\omega^2 + a\omega + b = 0$$

with $D = a^2 - 4b \geq 0, \omega < 0$ if and only if $a, b > 0$; implies that $\sigma^2 < 0$ if and only if

$$c_0^2 k^2 - 2\Omega_0^2 > 0.$$ 

Making an interpretation of these results, we can deduce that there will only be $\sigma^2 > 0$ and hence unstable behavior should

$$c_0^2 k^2 - 4\pi G_0 \rho_0 < 0,$$

which is usually referred to as Jeans’ gravitational instability criterion after Sir James Jeans [5] who first proposed it.
Jeans’ Instability Criterion

This instability criterion can be posed in terms of wavelength
\[ \lambda = \frac{2\pi}{k} \]
and then takes the form

\[ \lambda > \lambda_J = c_0 \sqrt{\frac{\pi}{G_0 \rho_0}} \equiv \text{Jeans’ length}, \]

where \( c_0 \equiv \text{speed of sound in an adiabatic gas of density } \rho_0 \).

This formula is of fundamental importance in astrophysics and cosmology where many significant deductions concerning the formation of galaxies and stars have been based upon it. In particular Jeans’ [5] interpretation of the criterion now bearing his name was that a gas cloud of characteristic dimension much greater than \( \lambda_J \) would tend to form condensations with mean distance of separation comparable to \( \lambda_J \) that then developed into those protostars observable in the outer arms of spiral galaxies such as Andromeda M31.
Figure: A Galaxy Evolution Explorer image of the Andromeda galaxy M31, courtesy of NASA/JPL-Caltech.
Comparisons with Jeans

Jeans [4] treated his static solution involving $\rho_0$ and $\varphi_0$ symbolically. Since Jeans’ original analysis was for a nonrotating system with $\Omega_0 = 0$, when he assumed in addition that $\rho_0$ was a constant in his perturbation equations to make them constant coefficient this implicitly required $\nabla \varphi_0 = 0$ which implied $\nabla^2 \varphi_0 = 0 = 4\pi G_0 \rho_0$ or $\rho_0 = 0$ and hence is termed Jeans’ swindle by Binney and Tremaine [1]. The problem under examination illustrates the point that adding rotation to the system as Chandrasekhar did and then performing a standard linear stability analysis of its exact static solution yields Jeans’ instability criterion but in a systematic manner and such a model also has the added advantage of being more astrophysically realistic.
Sekimura et al. [10] have demonstrated that for a secular equation similar in form to Jeans’ $\sigma^2 = 2\Omega_0^2 - c_0^2k^2$, $\lambda_J$ actually corresponds to the so-called critical wavelength $\lambda_c$ of linear stability theory associated with $\sigma = 0$ while nonlinear stability analyses of physical phenomena involving related secular equations have shown that the observed wavelengths are determined to a close approximation by this $\lambda_c$ rather than by the dominant wavelength $\lambda_d$ at which $\sigma$ achieves its maximum value from linear theory.
Jeans’ Secular Equation I

Figure: Schematic plots in the $k$-$\sigma$ plane depicting the methodology employed by Sekimura et al. [10] applied to the Jeans’ secular equation

$$\sigma = \sigma(k; c) = \sqrt{2\Omega_0^2 - c^2 k^2}.$$ That curve is plotted for both a general speed of sound $c$ and our specific speed $c_0 > c$ in this figure where $k_J = 2\pi/\lambda_J$ is such that $\sigma(k_J; c_0) = 0$. In a weakly nonlinear stability analysis one takes the disturbance wavenumber $k \equiv k_J$ and its growth rate to be equal to $\sigma_J(c) = \sigma(k_J; c) = \delta^2 > 0$ where $c$ is close enough to $c_0$ so that $\delta$ is a small parameter. Then in the $\lim_{c \to c_0} \sigma_J(c) = 0$ which is a requirement for the application of weakly nonlinear stability theory and any re-equilibrated pattern will exhibit a wavelength of $\lambda_J$. Here $c^2 = \gamma p_0 / \rho_0$ with $\gamma < \gamma_0$ and hence the operation $\lim_{c \to c_0}$ is equivalent to $\lim_{\gamma \to \gamma_0}$. 

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Jeans’ Secular Equation II

Therefore Jeans’ interpretation, although unusual for linear stability theory (where it is often presumed that such a disturbance associated with the largest growth rate predominates), both anticipated and is consistent with these nonlinear results, since, by the time perturbations have grown enough for the effect of the maximum growth rate to be observed, the neglected nonlinearities may have rendered that linear analysis inaccurate. In this context note that for a typical value of $\theta \neq \pi/2$, namely $\theta = 0$, we can factor our secular equation to obtain the roots $\sigma^2 = -4\Omega_0^2$ and $\sigma^2 = 2\Omega_0^2 - c_0^2k^2$. Observe that this last condition which yields our instability is equivalent to Jeans’ secular equation.
Jeans’ Length

Using the formula

\[ \lambda_J = c_0 \sqrt{\frac{\pi}{G_0 \rho_0}} \equiv \text{Jeans’ length} \]

with the parameters \( c_0 \) and \( \rho_0 \) assigned the values

\[ c_0 = \frac{2}{3} \times 10^4 \ \text{cm/sec} \quad \text{and} \quad \rho_0 = 10^{-22} \ \frac{\text{gm}}{\text{cm}^3} \]

employed by Jeans [5] for this purpose but when the polytropic index \( \gamma_0 = 4/3 \) while taking

\[ G_0 = 6.67 \times 10^{-8} \ \frac{\text{cm}^3}{\text{gm sec}^2} \]

in cgs units,
...yields

$$\lambda_J = 4.58 \times 10^{18} \text{ cm} = 1.48 \text{ pc}$$

where 1 pc $\equiv 3.09 \times 10^{18}$ cm, which compares quite favorably with the mean distance between actual adjacent condensations originally formed in the outer arms of Andromeda since, in those parts of M31, the averaged observed distance between protostars in such chains is about 1.4 pc or somewhat more if allowances are made for foreshortening [5].
Jeans’ Mass

Using the definition for Jeans’ mass as that contained in a sphere of diameter $\lambda_J$ \[1\], we find that

$$M_J = \frac{4\pi}{3} \left( \frac{\lambda_J}{2} \right)^3 \rho_0 = \frac{\pi}{6} \lambda_J^3 \rho_0 = 5 \times 10^{30} \text{ kgm},$$

which represents the mass surrounding each of these condensations as compared with the mass of the sun $M_\odot = 2 \times 10^{30} \text{ kgm}$ or exactly two and one-half times as much. This differs slightly from Jeans’ \[5\] original critical mass defined by

$$M_c = \lambda_J^3 \rho_0 = 9.55 \times 10^{30} \text{ kgm},$$

or about four and three-quarters times the mass of the sun.
**Figure**: A plot of size versus color for Population I stars found in the outer arms of spiral galaxies. Here the scale on the vertical axis indicates size in terms of solar masses and that on the horizontal, color from blue at the left to red at the right with yellow between them. The spiral arm stars show a simple color-size relationship: The largest ones are blue giants and the smallest, red dwarfs; while the sun-type of intermediate size is a so-called yellow dwarf.
Thus, Jeans got the right answer for the wrong reason. In his review of nonlinear hydrodynamic stability theory, the renowned comprehensive applied mathematical modeler Lee Segel [9] stated that

“All anyone can get the right answer for the right reason. It takes a genius or a physicist to get the right answer for the wrong reason.”

In this context, Sir James Jeans was both.
Comparisons with Chandrasekhar

Chandrasekhar’s [2] analysis with rotation added to Jeans’ perturbation equations also suffered from the same deficiency as the latter in that he did not develop a parameter relationship for his implicit exact solution and thus erroneously treated $\rho_0$ and $\Omega_0$ as independent. Besides Jeans’ criterion for $\theta \neq \pi/2$, this yielded an extraneous instability criterion for the case $\theta = \pi/2$; Namely: $c_0^2 k^2 < 4(\pi G_0 \rho_0 - \Omega_0^2)$ should $\Omega_0^2 < \pi \rho_0 G_0$. Chandrasekhar [2] plotted $\sigma^2$ versus $k$ for $\theta = 0, \pi/4, \pi/2$ and $\Lambda^2 \equiv \Omega_0^2/(\pi G_0 \rho_0) = 0.5, 1.0, 2.0$. In point of fact, $\Lambda^2 = 0.5$ is a representative value of that quantity for this extra instability condition while $\Lambda^2 = 2.0$, his upper bound, actually corresponds to its value as per our formula relating these parameters which violates that extra condition identically.
Chandrasekhar’s Extraneous Instability Criterion for \( \theta = \pi/2 \) and \( \Lambda^2 < 1.0 \)

**Figure:** Plots of
\[
\omega^2 = -\sigma^2 = c_0^2 k^2 + 4(\Omega^2 - \pi G \rho_0)
\]
versus \( k \) for Chandrasekhar’s dispersion relation when \( \theta = \pi/2 \) and \( \Lambda^2 < 1.0 \) or \( \Lambda^2 > 1.0 \), the lower or upper curves, respectively. Note that \( \Lambda^2 = 0.5 \) and 2.0 serve as representative values for these two cases, separated by \( \Lambda^2 = 1.0 \).
Comparison with Rubin and Ford

Let us examine the plausibility of our formula for

\[ \Omega_0 = \sqrt{2\pi \rho_0 G_0}. \]

In conjunction with the values for \( \rho_0 \) and \( G_0 \), this yields the uniform rotation

\[ \Omega_0 = 6.47 \times 10^{-15} \text{/sec} \]

and the corresponding rotational velocity

\[ V_0 = \Omega_0 R_0 = 200 \text{ km/sec}, \]
for the reference radial distance of

\[
R_0 = 1 \text{kpc} = 10^3 \text{pc} = 3.09 \times 10^{21} \text{cm} = 3.09 \times 10^{16} \text{km},
\]

both of which are consistent with the spectroscopic measurements of the Andromeda nebula and the observational data of the spiral Milky Way galaxy as reported by Rubin and Ford [7].
Rotation Curves for M31 and the Galaxy

Figure: Comparison of rotational velocities $V_0$ (km/sec) of Andromeda M31 and the Milky Way Galaxy as functions of radial distance to the center $R_0$ (kpc) where the solid and dashed curves represent these plots for M31 [7] and the Galaxy [8], respectively. The filled circles are the 7 observed rotational velocities for the Galaxy adopted by Roogour and Oort [6] to make the features of the latter conform with those of Andromeda.
Comparison with Binney and Tremaine

We close by noting that Binney and Tremaine [1] considered this gravitational instability model in a cylindrical rotating system as a problem in Chapter 5 of their book “Galactic Dynamics”. They observed that rotation allowed the Jeans’ instability to be analyzed exactly. Since the first part of their problem was to find the condition on $\Omega_0$ so that the homogeneous quiescent gas would be in equilibrium, Binney and Tremaine [1] did not examine the plausibility of this condition.
Further, the last part of their problem was to show, upon finding the resulting dispersion relation from its linear stability analysis, that waves propagating perpendicular to the rotation vector were always stable while those propagating parallel to it were unstable if and only if the usual Jeans’ criterion without rotation was satisfied. Although the latter conclusion for $\theta = 0$ agrees with our predictions, the former does not since, when $\theta = \pi/2$, we predicted $\sigma^2 = 0$ as well as those $\sigma^2 < 0$ which only implies a condition of neutral stability.
These results demonstrate that the best way to test the validity of a model for a natural science phenomenon is to compare its theoretical predictions with observable data of that phenomenon. Sir Arthur Conan Doyle characterized this philosophy perhaps as well as anyone by a Sherlock Holmes quote from A Scandal in Bohemia in his 1891 collection entitled The Adventures of Sherlock Holmes:

“It is a capital mistake to theorize before one has data. Insensibly one begins to twist facts to suit theories, instead of theories to suit facts.”
References I


References II


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Thank you!
Questions?