Proportion estimators, Lebesgue-Stieltjes integral equations, and concentration of measures
Outline

- Background on proportion estimation
- Brief review on existing methods
- The new method
- Summary
- Identify among hundreds of drugs those that may have a specific side effect
- Identify among thousands of genes those that might be associated with a disease
- Identify among millions of SNPs those that might be associated with the yield of a crop
Signals are entities that provide scientific information or that a scientist is searching for. Given a data set, one asks 4 basic questions:

1. Are there any signals? (Testing)
2. If there are signals, how many signals? (Testing)
3. If there are signals, where are they? (Testing)
4. If there are signals, what are their magnitudes? (Estimation)
Convention: “a signal” is identified with “a false null”, and “a noise” identified with “a true null”

Among $m$ null hypotheses, the proportion of false nulls, i.e., the signal proportion, is defined as

$$\pi_{1,m} = \frac{\text{Number of false nulls}}{m}$$

Example: in gene-disease association study, $\pi_{1,m}$ is the ratio of the number of disease-contributing genes to the total number of genes under investigation
Paradigms of sparsity

- Dense regime: $\liminf_{m \to \infty} \pi_{1,m} > 0$, i.e., there is always a positive proportion of signals

- Moderately sparse regime: $\pi_{1,m} = O(m^{-s})$ for $0 < s < 0.5$

- Critically sparse regime: $\pi_{1,m} = O(m^{-0.5})$

- Very sparse regime: $\pi_{1,m} = O(m^{-s})$ for $0.5 < s < 1$
• Data: \( \{z_i\}_{i=1}^m \) rv’s each with mean or median \( \mu_i \)

• Hypotheses: \( H_{i0} : \mu_i = \mu_0 \) vs \( H_{i1} : \mu_i \neq \mu_0 \)

• Indices of signals: \( I_{1,m} = \{1 \leq i \leq m : \mu_i \neq \mu_0\} \)

• Signal proportion: \( \pi_{1,m} = m_1 m^{-1} \), where \( m_1 = |I_{1,m}| \)

• Target: to consistently estimate \( \pi_{1,m} \)
A brief review on existing approaches

Four approaches to proportion estimation:

1. Hard thresholding: $H_{i0}$ is false when $|z_i| \geq t$ for some $t > 0$

2. Deconvolving two-component mixture model: $\{z_i\}_{i=1}^m$
   identically distributed as
   
   $f(x) = (1 - \pi_{1,m})f_0(x) + \pi_{1,m}f_1(x)$

3. Bounding an excess of uniform empirical process

4. Fourier transform, implemented for Gaussian family or mixtures with a Gaussian component
Disadvantages of the four approaches:

- Only Approaches 3 and 4 are consistent when $\pi_{1,m}$ tends to zero.
- Approach 4 is not applicable when CDFs of rv’s do not consist at least one component from a location-shift family.
- Consistency of Approach 3 requires CDFs to be absolutely continuous.
- Consistency of Approaches 1 and 2 require restrictive modelling assumptions (e.g., $z_i$’s corresponding to false nulls are identically distributed).
The strategy: set-up

- Let \( \mathbf{z} = (z_1, \ldots, z_m) \) and \( \bm{\mu} = (\mu_1, \ldots, \mu_m) \)

- Denote by \( F_{\mu_i} \) the CDF of \( z_i \) such that \( F_{\mu_i} \in \mathcal{F} = \{F_\mu : \mu \in U\} \), where \( U \) has a non-empty interior

- Each \( F_\mu \) is uniquely determined by \( \mu \)

- Recall: \( H_{i0} : \mu_i = \mu_0 \quad \text{vs} \quad H_{i1} : \mu_i \neq \mu_0 \)
The strategy: constructing “Phase function”

- For each $\mu \in U$, approximate $1_{\{\mu \neq \mu_0\}}$ by $\psi (t, \mu; \mu_0)$, $t \in \mathbb{R}$ such that

  $$\lim_{t \to \infty} \psi (t, \mu_0; \mu_0) = 1 \text{ and } \lim_{t \to \infty} \psi (t, \mu; \mu_0) = 0 \text{ for } \mu \neq \mu_0$$  

(1)

- Define “phase function”

  $$\varphi_m (t, \mu) = \frac{1}{m} \sum_{i=1}^{m} [1 - \psi (t, \mu_i; \mu_0)]$$  

(2)

- The phase function: $\lim_{t \to \infty} \varphi_m (t, \mu) = \pi_{1,m}$ for any fixed $m$ and $\mu$, i.e., the “Oracle”

  $$\Lambda_m (\mu) = \lim_{t \to \infty} \varphi_m (t, \mu)$$  

(3)
The strategy: constructing “Empirical phase function”

- Construct $K : \mathbb{R}^2 \to \mathbb{R}$ that is independent of $\mu \neq \mu_0$ and solves the Lebesgue-Stieltjes integral equation

$$\psi(t, \mu; \mu_0) = \int K(t, x; \mu_0) \, dF_\mu(x)$$  \hspace{1cm} (4)

- Define “empirical phase function”

$$\hat{\phi}_m(t, z) = \frac{1}{m} \sum_{i=1}^{m} \left[ 1 - K(t, z_i; \mu_0) \right]$$  \hspace{1cm} (5)

- The empirical phase function: for any fixed $m, t$ and $\mu$

$$\mathbb{E}[\hat{\phi}_m(t, z)] = \varphi_m(t, \mu)$$  \hspace{1cm} (6)
The strategy: intuition for consistency

- Recall “phase function” \( \varphi_m(t, \mu) \) and \( \lim_{t \to \infty} \varphi_m(t, \mu) = \pi_{1,m} \)
- Recall “empirical phase function” \( \hat{\varphi}_m(t, z) \) and
  \[
  \mathbb{E} [\hat{\varphi}_m(t, z)] = \varphi_m(t, \mu)
  \]
- Intuition: \( \hat{\varphi}_m(t, z) \) will consistently estimate \( \pi_{1,m} \), i.e.,
  \[
  \Pr \left( \left| \frac{\hat{\varphi}_m(t_m, z)}{\pi_{1,m}} - 1 \right| \to 0 \right) \to 1 \quad \text{as} \quad m \to \infty \tag{7}
  \]
  if
  \[
  e_m(t_m) = \left| \hat{\varphi}_m(t_m, z) - \varphi_m(t_m, \mu) \right| \to 0 \text{ at appropriate speed}
  \]
Phase functions may not exist, i.e., no solution in $\psi$ to

$$\lim_{t \to \infty} \psi(t, \mu_0; \mu_0) = 1 \text{ and } \lim_{t \to \infty} \psi(t, \mu; \mu_0) = 0 \text{ for } \mu \neq \mu_0$$

or in $K$ (independent of $\mu \neq \mu_0$) to

$$\psi(t, \mu; \mu_0) = \int K(t, x; \mu_0) \, dF_\mu(x)$$

The oscillation

$$e_m(t_m) = \left| \hat{\phi}_m(t_m, z) - \varphi_m(t_m, \mu) \right|$$

may not converge to 0 when $K$ is not a Lipschitz or bounded transform of $\{z_i\}_{i=1}^m$. 

The strategy: main difficulties in implementation
When \( \{z_i\}_{i=1}^m \) are mutually independent, uniformly consistent estimators \( \hat{\phi}_m(t, z) \) are constructed respectively for the estimation problem in the settings where

- Random variables have Riemann-Lebesgue type characteristic functions (RL Type CFs)

- Random variables have CDFs that are members of natural exponential families with support \( \mathbb{N} \)

- Random variables have CDFs that are members of natural exponential families with separable moments
Construction I: families with Riemann-Lebesgue type CFs

- Recall $\mathcal{F} = \{F_\mu : \mu \in U\}$ and let $\hat{F}_\mu(t) = \int e^{itx} dF_\mu(x)$ be the CF of $F_\mu$ where $i = \sqrt{-1}$
- Let $r_\mu$ be the modulus of $\hat{F}_\mu$, so that $\hat{F}_\mu = r_\mu e^{ih_\mu}$

**Definition (Riemann-Lebesgue type CF)**

If $\{t \in \mathbb{R} : \hat{F}_{\mu_0}(t) = 0\} = \emptyset$, $\sup_{t \in \mathbb{R}} \frac{r_\mu(t)}{r_{\mu_0}(t)} < \infty$ for each $\mu \in U \setminus \{\mu_0\}$, and

$$\lim_{t \to \infty} \frac{1}{t} \int_{[-t,t]} \frac{\hat{F}_\mu(y)}{\hat{F}_{\mu_0}(y)} dy = 0,$$  \hspace{1cm} (8)

then $\hat{\mathcal{F}} = \{\hat{F}_\mu : \mu \in U\}$ is said to be of “Riemann-Lebesgue type (RL type)” at $\mu_0$ on $U$. 

Let $\omega : [-1, 1] \to \mathbb{R}_+$ be bounded and Lebesgue-integrates to 1

**Theorem (RVs with RL Type CFs)**

Let $\hat{\mathcal{F}}$ be of RL type and define $K : \mathbb{R}^2 \to \mathbb{R}$ as

$$K(t, x; \mu_0) = \int_{[-1, 1]} \frac{\omega(s) \cos(tsx - h_{\mu_0}(ts))}{r_{\mu_0}(ts)} ds. \quad (9)$$

Then

$$\psi(t, \mu; \mu_0) = \int_{[-1, 1]} \omega(s) \frac{r_{\mu}(ts)}{r_{\mu_0}(ts)} \cos(h_{\mu}(ts) - h_{\mu_0}(ts)) ds. \quad (10)$$
Construction I: application to location-shift families

\[ \mathcal{F} = \{ F_\mu : \mu \in U \} \] is a location-shift family if and only if \( z + \mu' \) has CDF \( F_{\mu + \mu'} \) whenever \( z \) has CDF \( F_\mu \) for \( \mu, \mu + \mu' \in U \)

Corollary (Construction I for Location-shift families)

If \( \mathcal{F} \) is a location-shift family for which \( \{ t \in \mathbb{R} : \hat{F}_{\mu_0}(t) = 0 \} = \emptyset \), then

\[
\psi(t, \mu; \mu_0) = \int K(t, y + (\mu - \mu_0); \mu_0) \, dF_{\mu_0}(y)
\]

\[
= \int_{[-1,1]} \omega(s) \cos(ts(\mu - \mu_0)) \, ds.
\]
Construction I: some examples

- **Gaussian family** \( \text{Normal} (\mu, \sigma^2) \) with mean \( \mu \) and std dev \( \sigma > 0 \), for which
  \[
  K(t, x; \mu_0) = \int_{[-1,1]} \exp \left( 2^{-1} t^2 s^2 \sigma^2 \right) \omega(s) \exp (\imath ts (x - \mu_0)) \, ds
  \]

- **Laplace family** \( \text{Laplace} (\mu, 2\sigma^2) \) with mean \( \mu \) and std dev \( \sqrt{2}\sigma > 0 \), for which
  \[
  K(t, x; \mu_0) = \int_{[-1,1]} \left( 1 + \sigma^2 t^2 s^2 \right) \omega(s) \exp (\imath ts (x - \mu_0)) \, ds
  \]

- **Cauchy family** \( \text{Cauchy} (\mu, \sigma) \) with median \( \mu \) and scale parameter \( \sigma > 0 \), for which
  \[
  K(t, x; \mu_0) = \int_{[-1,1]} \exp (\sigma |ts|) \omega(s) \exp (\imath ts (x - \mu_0)) \, ds
  \]
Uniform consistency classes

Definition (Uniform consistency class)

Given a family $\mathcal{F}$, the sequence of sets $Q_m(\mu, t; \mathcal{F}) \subseteq \mathbb{R}^m \times \mathbb{R}$ for each $m \in \mathbb{N}_+$ is called a “uniform consistency class” for the estimator $\hat{\phi}_m(t, z)$ if

$$\Pr\left(\sup_{\mu \in Q_m(\mu, t; \mathcal{F})} \left| \pi_{1,m}^{-1} \sup_{t \in Q_m(\mu, t; \mathcal{F})} \hat{\phi}_m(t, z) - 1 \right| \to 0 \right) \to 1.$$  

- Define

$$B_m(\rho) = \{ \mu \in \mathbb{R}^m : m^{-1} \sum_{i=1}^m |\mu_i - \mu_0| \leq \rho \}$$  

for some $\rho > 0$

and $u_m = \min \{|\mu_j - \mu_0| : \mu_j \neq \mu_0\}$
Construction I: applied to location-shift families

Theorem (Uniform consistency for Construction I)

If $\mathcal{F}$ is a “good” location-shift family, then a uniform consistency class is

$$
\mathcal{Q}_m(\mu, t, \mathcal{F}) = \left\{ R_m(\rho) = O\left( m^{\delta'} \right), \tau_m \leq \gamma_m, u_m \geq \frac{\log \log m}{\gamma' \tau_m}, \right. \\
\left. t \in [0, \tau_m], \lim_{m \to \infty} \pi_{1,m}^{-1} \Upsilon (q, \tau_m, \gamma_m, r_{\mu_0}) = 0 \right\},
$$

where $R_m(\rho) \sim \rho$, $\gamma_m = \gamma' \log m$

$$
\Upsilon (q, \tau_m, \gamma_m, r_{\mu_0}) = \frac{2 \| \omega \|_{\infty} \sqrt{2q\gamma_m}}{\sqrt{m}} \sup_{t \in [0, \tau_m]} \int_{[0,1]} ds \left\| r_{\mu_0}(ts) \right\|.
$$

Moreover, for all sufficiently large $m$, with probability at least $1 - o(1)$,

$$
\sup_{\mu \in \mathcal{B}_m(\rho)} \sup_{t \in [0, \tau_m]} |\hat{\varphi}_m(t, z) - \varphi_m(t, \mu)| \leq \Upsilon (q, \tau_m, \gamma_m, r_{\mu_0}).
$$
For location-shift families, recall $r_{\mu_0}$ as the modulus of CF $\hat{F}_{\mu_0}$ for CDF $F_{\mu_0}$ and

$$
\Upsilon(q, \tau_m, \gamma_m, r_{\mu_0}) = \frac{2 \| \omega \|_{\infty} \sqrt{2 q \gamma_m}}{\sqrt{m}} \sup_{t \in [0, \tau_m]} \int_{[0,1]} ds \ r_{\mu_0}(ts)
$$

- The speed of convergence and uniform consistency class for the estimator $\hat{\phi}_m(t, z)$ is mainly determined by $r_{\mu_0}$ (not by $r_{\mu}$ with $\mu \neq \mu_0$) and

  $$
u_m = \min \{|\mu_j - \mu_0| : \mu_j \neq \mu_0\}$$

- Uniform consistency can be achieved when $\{z_i\}_{i=1}^m$ do not have bounded first-order absolute moment (e.g., Cauchy family)

- It is unlikely that $\hat{\phi}_m(t, z)$ can be consistent for $\pi_{1,m} = O(m^{-0.5})$
A brief review on Natural Exponential Family (NEF)

- Let $\beta$ be a positive non-Dirac Radon measure on $\mathbb{R}$
- Let $L(\theta) = \int e^{x\theta} \beta(dx)$ for $\theta \in \mathbb{R}$ and $\Theta$ be the maximal open set containing $\theta$ such that $L(\theta) < \infty$
- Assume $\Theta$ is not empty and let $\kappa(\theta) = \log L(\theta)$. Then

$$\mathcal{F} = \{G_\theta : G_\theta(dx) = \exp(\theta x - \kappa(\theta)) \beta(dx), \theta \in \Theta\}$$

forms an NEF with respect to the basis $\beta$

- Fact: $\Theta$ is convex if it has a non-empty interior; $L$ is analytic on the strip $A_\Theta = \{z \in \mathbb{C} : \Re(z) \in \Theta\}$

- Fact: $\mathcal{F}$ can be equivalently characterized as $\mathcal{F} = \{F_\mu : \mu \in U\}$, where $\mu : \Theta \to U$ is the mean function with $U = \mu(\Theta)$
Constructions II and III: notational set-up

- $\mu_0$ is identified with $\theta_0$, and $\mu = \mu(\theta)$ for $\theta \in \Theta$
- Reuse $\psi$ but take it as a function of $t$ and $\theta$
- Reuse $K$ but take it as a function of $t$ and $\theta$, such that

$$\psi(t, \theta; \theta_0) = \int K(t, x; \theta_0) \, dG_\theta(x) \text{ for } G_\theta \in \mathcal{F}.$$ 

- Let $\theta = (\theta_1, \ldots, \theta_m)$, the phase function

$$\varphi_m(t, \theta) = \frac{1}{m} \sum_{i=1}^{m} \left[ 1 - \psi(t, \theta_i; \theta_0) \right]$$

and the empirical phase function

$$\hat{\varphi}_m(t, z) = \frac{1}{m} \sum_{i=1}^{m} \left[ 1 - K(t, z_i; \theta_0) \right].$$
Constructions II and III: differences from Construction I

Outside families with Riemann-Lebesgue type CFs

- Fourier transform used to construct $K$ is no longer applicable
- $K$ does not always exist as a solution to

$$
\psi (t, \theta; \theta_0) = \int K(t, x; \theta_0) \, dG_\theta (x) \text{ for } G_\theta \in \mathcal{F}
$$

- Even if $K$ does exist, the transform induced by $K$ on $\{z_i\}_{i=1}^m$ is no longer Lipschitz or bounded, which causes great difficulty in deriving concentration inequalities for the oscillation

$$
e_m(t_m) = \left| \hat{\phi}_m(t_m, z) - \varphi_m(t_m, \mu) \right|
$$

- Special functions and their asymptotics are needed since NEFs are often related to these functions
Assume the basis $\beta$ for $\mathcal{F}$ is discrete with support $\mathbb{N}$, i.e., there exists a positive sequence $\{c_k\}_{k \geq 0}$ such that

$$\beta = \sum_{k=0}^{\infty} c_k \delta_k$$

- The power series $H(z) = \sum_{k=0}^{\infty} c_k z^k$ with $z \in \mathbb{C}$ must have a positive radius of convergence $R_H$

- $h$ is the generating function (GF) of $\beta$

- If $\beta$ is a probability measure, then $(-\infty, 0] \subseteq \Theta$ and $R_H \geq 1$, and vice versa
Theorem (Construction for discrete NEF with support $\mathbb{N}$)

Let $\mathcal{F}$ be an NEF generated by $\beta$ with support $\mathbb{N}$. For $x \in \mathbb{N}$ and $t \in \mathbb{R}$ set

$$K(t, x; \theta_0) = H(e^{\theta_0}) \int_{[-1,1]} \frac{(ts)^x \cos \left( \frac{\pi x}{2} - tse^{\theta_0} \right)}{H(x)(0)} \omega(s) \, ds. \quad (11)$$

Then

$$\psi(t, \theta; \theta_0) = \int K(t, x; \theta_0) dG_\theta(x)$$

$$= \frac{H(e^{\theta_0})}{H(e^{\theta})} \int_{[-1,1]} \cos \left( st \left( e^{\theta} - e^{\theta_0} \right) \right) \omega(s) \, ds. \quad (12)$$
• Poisson family Poisson \((\mu)\) with mean \(\mu > 0\). The basis

\[
\beta = \sum_{k=0}^{\infty} \frac{1}{k!} \delta_k \quad \text{with} \quad L(\theta) = \exp(e^\theta) \quad \text{for} \quad \theta \in \mathbb{R},
\]

\[
\mu(\theta) = e^\theta, \quad H(z) = e^z \quad \text{with} \quad R_H = \infty \quad \text{and} \quad H^{(k)}(0) = 1 \quad \text{for all} \quad k \in \mathbb{N}
\]

• Negative Binomial family NegBinomial \((\theta, n)\) with \(\theta < 0\) and \(n \in \mathbb{N}_+\). The basis is

\[
\beta = \sum_{k=0}^{\infty} \frac{c^*_k}{k!} \delta_k \quad \text{with} \quad c^*_k = \frac{(k+n-1)!}{(n-1)!} \quad \text{for} \quad k \in \mathbb{N},
\]

\[
L(\theta) = \left(1 - e^\theta\right)^{-n} \quad \text{with} \quad \theta < 0, \quad \mu(\theta) = ne^\theta \left(1 - e^\theta\right)^{-1},
\]

\[
H(z) = (1 - z)^{-n} \quad \text{with} \quad R_H = 1, \quad \text{and} \quad H^{(k)}(0) = c^*_k \quad \text{for all} \quad k \in \mathbb{N}
\]
Construction II: uniform consistency classes

Set $\eta = e^{\theta}$ for $\theta \in \Theta$ and $\eta = (e^{\theta_1}, \ldots, e^{\theta_m})$.

Theorem (Uniform consistency for Construction II)

Let $\mathcal{F}$ be an NEF generated by $\beta$ with support $\mathbb{N}$ and $\rho$ a finite, positive constant. If $\frac{1}{c_k k!} \leq \frac{C_k}{k!}$ for all $k \in \mathbb{N}$, then a uniform consistency class is

$$
\mathcal{Q}_{II,1}(\theta, t, \pi_{1,m}; \gamma) = \left\{ \begin{array}{l}
\|\theta\|_{\infty} \leq \rho, \pi_{1,m} \geq m^{(\gamma-1)/2}, \\
t = 2^{-1} \|\eta\|_{\infty}^{-1/2} \gamma \log m, \\
\lim_{m \to \infty} t \min_{i \in I_{1,m}} |\eta_0 - \eta_i| = \infty
\end{array} \right. \\
for any fixed $\gamma \in (0, 1]$.
$$
• The GF of Poisson family has a constant derivative sequence at 0: $H^{(k)}(0) = c_k k! = 1$ for all $k \in \mathbb{N}$

Theorem (Uniform consistency for Construction II)

For Poisson family, a uniform consistency class is

$$\mathcal{D}_{II,2}(\theta, t, \pi_1, m; \gamma) = \begin{cases} \|\theta\|_{\infty} \leq \rho, \pi_1, m \geq m(\gamma' - 1)/2, \\ t = \sqrt{\|\eta\|_{\infty}^{-1/2}} \gamma \log m, \\ \lim_{m \to \infty} t \min_{i \in I_1, m} |\eta_0 - \eta_i| = \infty \end{cases}$$

for any fixed $\gamma \in (0, 1)$ and $\gamma' > \gamma$. 
Assume $0 \in \Theta$, so that $\beta$ is a probability measure with finite moments of all orders.

Let

$$\tilde{c}_n(\theta) = \frac{1}{L(\theta)} \int x^n e^{\theta x} \beta(dx) = \int x^n dG_\theta(x) \quad \text{for} \quad n \in \mathbb{N} \quad (13)$$

be the moment sequence for $G_\theta \in \mathcal{F}$.

Fact: (13) is the Mellin transform of the measure $G_\theta$.

Caution: Mellin transform is not Fourier transform even though they are connected.
Definition (NEF with separable moments)

If there exist two functions $\zeta, \xi : \Theta \to \mathbb{R}$ and a sequence $\{\tilde{a}_n\}_{n \geq 0}$ that satisfy the following:

- $\xi(\theta) \neq \xi(\theta_0)$ whenever $\theta \neq \theta_0$, $\xi(\theta) \neq 0$ for all $\theta \in U$, and $\zeta$ does not depend on any $n \in \mathbb{N}$,
- $\tilde{c}_n(\theta) = \xi^n(\theta) \zeta(\theta) \tilde{a}_n$ for each $n \in \mathbb{N}$ and $\theta \in \Theta$,
- $\Psi(t, \theta) = \sum_{n=0}^{\infty} \frac{t^n \xi^n(\theta)}{\tilde{a}_n n!}$ is absolutely convergent pointwise in $(t, \theta) \in \mathbb{R} \times \Theta$,

then the moment sequence $\{\tilde{c}_n(\theta)\}_{n \geq 0}$ is called “separable” (at $\theta_0$).
Theorem (Construction for NEFs with Separable Moments)

Assume that the NEF $F$ has a separable moment sequence $\{\tilde{c}_n(\theta)\}_{n \geq 0}$ at $\theta_0$. For $t, x \in \mathbb{R}$ set

$$K(t, x; \mu_0) = \frac{1}{\zeta(\theta_0)} \int_{[-1,1]} \sum_{n=0}^{\infty} \frac{(-tsx)^n \cos\left(\frac{n}{2} + ts\xi(\theta_0)\right)}{\tilde{a}_n n!} \omega(s) \, ds. \quad (14)$$

Then

$$\psi(t, \mu; \mu_0) = \int K(t, x; \theta_0) dG_\theta(x) = \frac{\zeta(\theta)}{\zeta(\theta_0)} \int_{[-1,1]} \cos(ts(\xi(\theta_0) - \xi(\theta))) \omega(s) \, ds. \quad (15)$$
Let $\Gamma$ be Euler’s Gamma function. Gamma family $\Gamma(\theta, \sigma)$:

- Its basis $\frac{d\beta}{d\nu}(x) = \frac{x^{\sigma-1}e^{-x}}{\Gamma(\sigma)} 1_{(0,\infty)}(x) \, dx$ with $\sigma > 0$

- $L(\theta) = (1 - \theta)^{-\sigma}$ and $\mu(\theta) = \frac{\sigma}{1-\theta}$

- CDF and PDF:

  $$\frac{dG_\theta}{d\nu}(x) = f_\theta(x) = (1 - \theta)^\sigma \frac{e^{\theta x} x^{\sigma-1} e^{-x}}{\Gamma(\sigma)} 1_{(0,\infty)}(x) \text{ for } \theta < 1$$

- The Gamma family subsumes Exponential family and central Chi-square family
Construction III: one example

- Fact:

\[ \tilde{c}_n(\theta) = (1 - \theta)^\sigma \int_0^\infty e^{-y} y^{n+\sigma-1} \frac{dy}{\Gamma(\sigma)} \frac{\Gamma(n+\sigma)}{(1-\theta)^{n+\sigma-1}} = \frac{\Gamma(n+\sigma)}{\Gamma(\sigma)} \frac{1}{(1-\theta)^n}, \]

which implies \( \xi(\theta) = (1 - \theta)^{-1} \), \( \tilde{a}_n = \frac{\Gamma(n+\sigma)}{\Gamma(\sigma)} \) and \( \zeta \equiv 1 \)

- Construction III:

\[ K(t, x; \theta_0) = \int_{[-1,1]} \omega(s) \sum_{n=0}^{\infty} (-txs)^n \frac{\Gamma(n) \cos\left(\frac{\pi}{2} n + \frac{ts}{1-\theta_0}\right)}{n! \Gamma(\sigma + n)} ds, \]

for which

\[ \psi(t, \theta; \theta_0) = \int_{[-1,1]} \cos\left(ts\left((1-\theta_0)^{-1} - (1-\theta)^{-1}\right)\right) \omega(s) ds \]
Recall Construction I applied to location-shift family:

\[ \psi(t, \mu; \mu_0) = \int K(t, x; \mu_0) \, dF_{\mu}(x) \]
\[ = \int K(t, y + (\mu - \mu_0); \mu_0) \, dF_{\mu_0}(y), \]

where Fourier transform: \( K(t, x; \mu_0) \mapsto K(t, y + (\mu - \mu_0); \mu_0) \)

Construction III for Gamma family:

\[ \psi(t, \theta; \theta_0) = \int K(t, x; \theta_0) \, dG_{\theta}(x) \]
\[ = \int K(t, \frac{y}{1 - \theta}; \theta_0) \, \beta(dy), \]

where Mellin transform: \( K(t, x; \theta_0) \mapsto K(t, y (1 - \theta)^{-1}; \theta_0) \)
Construction III: uniform consistency class

- Set $u_{3,m} = \min_{1 \leq i \leq m} \{1 - \theta_i\}$ and $\xi(\theta) = (1 - \theta)^{-1}$

Theorem (Uniform Consistency of Construction III)

Consider Gamma family with a fixed $\sigma > 0$. Let $\rho > 0$ be a finite constant. If $\sigma > 3/4$, then a uniform consistency class is

$\mathcal{Q}_{III}(\theta, t, \pi_1, m; \gamma) = \begin{cases} 
\|\theta\| \leq \rho, t = 4^{-1} \gamma u_{3,m} \log m, \\
\lim_{m \to \infty} u_{3,m} \log m = \infty, \\
\pi_{1,m} \geq m^{(\gamma^{-1})/2}, \\
\lim_{m \to \infty} t \min_{i \in I_1, m} |\xi(\theta_0) - \xi(\theta_i)| = \infty 
\end{cases}$

for any fixed $\gamma \in (0, 1]$. 
Theorem (Uniform Consistency of Construction III)

Consider Gamma family with a fixed $\sigma > 0$. Let $\rho > 0$ be a finite constant. If $\sigma \leq 3/4$, then

$$Q_{III}(\theta, t, \pi_1, m; \gamma) = \begin{cases} 
\|\theta\| \leq \rho, t = 4^{-1} \gamma u_{3,m} \log m, \\
\pi_{1,m} \geq m^{(\gamma' - 1)/2}, \\
\lim_{m \to \infty} t \min_{i \in I_{1,m}} |\xi(\theta_0) - \xi(\theta_i)| = \infty 
\end{cases}$$

for any fixed $\gamma \in (0, 1)$ and $\gamma' > \gamma$. 
Summary

- Solutions to Lebesgue-Stieltjes integral equation can serve as universal construction for proportion estimators
- Under independence, the induced estimators are often uniformly consistent
- The uniform consistency class and speed of convergence for each such estimator are largely determined by
  - the modulus of the CF corresponding to the null parameter for families with Riemann-Lebesgue type CFs (including many location-shift families)
  - the radius of convergence of the generating function of discrete natural exponential families
  - the decaying speed of the moment sequence of continuous natural exponential families
A few remaining tasks

- Construct estimators via solutions to Lebesgue-Stieltjes integral equations for $\pi_{1,m}$ for composite null hypotheses such as
  
  $H_{i0}: \mu_i \in (a, b)$ vs $H_{i1}: \mu_i \notin (a, b)$
  
  $H_{i0}: \mu_i \leq a$ vs $H_{i1}: \mu_i > a$

- Uniform consistency of such estimators under dependence outside Gaussian families and their associated mixtures

- Classify natural exponential families with separable moment functions
Classify distributions whose CFs satisfy

\[ \{ t \in \mathbb{R} : \hat{F}_\mu(t) = 0 \} = \emptyset \quad \text{for} \quad \mu \in U' \]  

(16)

Completely understand the relationships between families with Riemann-Lebesgue type CFs, those whose CFs satisfy (16), and infinitely divisible distributions.

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References in the above preprint