

Estimators of the proportion of false null hypotheses: I “universal construction via Lebesgue-Stieltjes integral equations and uniform consistency under independence”

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Abstract

The proportion of false null hypotheses is a very important quantity in statistical modeling and inference based on the two-component mixture model and its extensions, and in control and estimation of the false discovery rate and false non-discovery rate. Most existing estimators of this proportion threshold p-values, deconvolve the mixture model under constraints on its components, or depend heavily on the location-shift property of distributions. Hence, they usually are not consistent, applicable to non-location-shift distributions, or applicable to discrete statistics or p-values. To eliminate these shortcomings, we construct uniformly consistent estimators of the proportion as solutions to Lebesgue-Stieltjes integral equations. In particular, we provide such estimators respectively for random variables whose distributions have Riemann-Lebesgue type characteristic functions, form discrete natural exponential families with infinite supports, and form natural exponential families with separable moment sequences. We provide the speed of convergence and uniform consistency class for each such estimator under independence. In addition, we provide example distribution families for which a consistent estimator of the proportion cannot be constructed using our techniques.

Keywords: Analytic functions; Bessel functions; concentration inequalities; Fourier transform; Lambert W functions; Lebesgue-Stieltjes integral equations; Mellin transform; natural exponential family; proportion of false null hypotheses

MSC 2010 subject classifications: Primary 35C05, 60E05; Secondary 60F10.

1 Introduction

The proportion of false null hypotheses and its dual, the proportion of true null hypotheses, play important roles in statistical modelling and multiple hypotheses testing. For example, they are components of the two-component mixture model of [Efron et al. \(2001\)](#), its extensions by [Ploner et al. \(2006\)](#), [Cai and Sun \(2009\)](#) and [Liu et al. \(2016\)](#), and their induced statistics including the “local false discovery rate (local FDR)” of [Efron et al. \(2001\)](#) and “positive FDR (pFDR)”

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and q-value of [Storey \(2003\)](#). They also form the optimal discovery procedure (ODP) of [Storey \(2007\)](#). However, without information on the proportions, decision rules based on the local FDR cannot be implemented, and none of the pFDR, q-value and ODP decision rule can be computed in practice. On the other hand, these proportions form upper bounds on the FDRs and false non-discovery rates (FNRs) of all FDR procedures including those of [Benjamini and Yekutieli \(2001\)](#), [Genovese and Wasserman \(2004\)](#), [Storey et al. \(2004\)](#), [Sarkar \(2006\)](#) and [Blanchard and Roquain \(2009\)](#). Therefore, accurate information on either proportion helps better control and estimate the FDR and FNR, and thus potentially enables an adaptive procedure to be more powerful or have smaller FNR than its non-adaptive counterpart. Since neither proportion is known in practice, it is very important to accurately estimate the proportions.

In this work, we focus on consistently estimating the proportions without employing the two-component mixture model, requiring p-values be identically distributed under the alternative hypotheses, or assuming that statistics or p-values have absolutely continuous cumulative distribution functions (CDFs). The main motivation for dealing with discrete p-values or statistics is the wide practice of FDR control in multiple testing based on discrete data in genomics ([Auer and Doerge; 2010](#); [Robinson and Smyth; 2008](#)), genetics ([Gilbert; 2005](#)), vaccine efficacy studies ([Mehrotra and Heyse; 2004](#)), drug safety monitoring ([Chen and Doerge; 2017](#)) and other areas, where Binomial Test (BT), Fisher’s Exact Test (FET) and Exact Negative Binomial Test (NBT) have been routinely used to test individual hypotheses. In the sequel, we refer to an estimator of the proportion of true (or false) null hypotheses as a “null (or alternative) proportion estimator”.

1.1 A brief review on existing proportion estimators

There are many estimators of the proportions, and their constructions can be roughly categorized into 4 classes: (1) thresholding p-values ([Storey; 2002](#); [Storey et al.; 2004](#)); (2) deconvolving the two-component mixture model for the distribution of p-values or statistics, modulo identifiability conditions ([Swanepoel; 1999](#); [Genovese and Wasserman; 2004](#); [Kumar and Bodhisattva; 2016](#)); (3) bounding the proportions via the use of uniform empirical process ([Meinshausen and Rice; 2006](#)); (4) Fourier transform for Gaussian family or mixtures with a Gaussian component ([Jin and Cai; 2007](#); [Jin; 2008](#); [Jin et al.; 2010](#)). Further, perhaps the only existing consistent proportion estimators are those mentioned right above, and only those in class (3) and (4) are consistent when the proportion tends to zero (see (5) for the definition of consistency). However, they have the following disadvantages: (i) the construction in class (4) based on Fourier transform does not work if the family of distributions of statistics does not consist at least one component from a location-shift family; (ii) consistency of estimators in class (1) and (2) requires the two-component mixture model and various regularity conditions such as concavity, smoothness, purity (defined by [Genovese and Wasserman; 2004](#)) of the alternative component, or the conditions in Lemma 3 or Lemma 4 of [Kumar and Bodhisattva \(2016\)](#); (iii) consistency

of the estimator in class (3) requires p-values to have continuous distributions, be independent, and be identically distributed under the alternative hypotheses.

In practice, the requirements needed by the consistent proportion estimators mentioned above are often not met. For example, for multiple testing where each statistic follows a Chi-square or Binomial distribution the involved distributions do not form a location-shift family, the two-component mixture model is inappropriate for p-values of BTs, FETs or NBTs with different marginal counts, and p-values under the alternative hypotheses cannot be identically distributed when the signal levels are different across individual hypotheses under the alternative. Further, almost all existing proportion estimators were initially designed for p-values or statistics that have continuous distributions, and it is not clear yet whether the null proportion estimators in classes (2) to (4) are reciprocally conservative even though the estimator of [Storey et al. \(2004\)](#) is so for independent p-values whose null distributions are uniform on $[0, 1]$ (see Corollary 13 of [Blanchard and Roquain; 2009](#)). Therefore, it is important to develop new, consistent proportion estimators that mitigate or eliminate the shortcomings of existing ones.

1.2 Main contributions and summary of results

We generalize “Jin’s Strategy” provided in Section 2.1 of [Jin \(2008\)](#) that estimates the proportions by approximating the indicator function of the true status of a null hypothesis and has only been implemented for Gaussian family or mixtures with a Gaussian component, and in much more general settings construct proportion estimators as solutions to a specific type of Lebesgue-Stieltjes integral equation in the complex domain. Generalizing Jin’s Strategy to non-location-shift families is highly nontrivial since this requires solving Lebesgue-Stieltjes integral equations and outside location-shift families solutions to these equations are very hard to find or do not exist. Even after a solution is found, to prove the uniform consistency of the proposed estimator we have to derive concentration inequalities for non-Lipschitz transforms of independent, unbounded random variables, and this is highly nontrivial and an open area in probability theory.

Our methodology (referred to as “the Strategy”) lifts Jin’s strategy to its full generality. In addition to the key advantage of Jin’s strategy, i.e., being independent from but partially applicable to the two-component mixture model, the Strategy applies to random variables that have continuous or discrete distributions or form non-location-shift families. Further, it produces uniformly consistent proportion estimators in the dense and moderately sparse regimes.

Our main contributions are summarized as follows:

- “Construction I”: Construction of uniformly consistent proportion estimators for random variables whose distributions have Riemann-Lebesgue type characteristic functions (RL type CFs); see [Section 3](#). In particular, this covers the construction for random variables from several location-shift families, including that in [Jin \(2008\)](#) for Gaussian and Laplace distributions.

- “Construction II”: Construction of uniformly consistent proportion estimators for random variables whose distributions form natural exponential families with infinite supports; see [Section 4](#) and [Section 6](#). In particular, this covers six of the twelve natural exponential families with cubic variance functions (NEF-CVFs) proposed by [Letac and Mora \(1990\)](#) such as Poisson and Negative Binomial families. Estimators from Construction II are perhaps the first consistent estimators that are applicable to discrete random variables.
- “Construction III”: Construction of uniformly consistent proportion estimators for random variables whose distributions form natural exponential families with separable moment sequences; see [Section 4](#) and [Section 7](#). In particular, this covers Gamma family which includes Exponential and Chi-square families as special cases.
- For each constructed estimator, its speed of convergence and uniform consistency class is provided under independence; see [Section 3](#), [Section 6](#) and [Section 7](#).

Specifically, Construction I employs Fourier transform and an extended Riemann-Lebesgue Lemma of [Costin et al. \(2016\)](#). Since the set of distributions with RL type CFs contains several location-shift families, we give a unified treatment of proportion estimators for location-shift families and reveal the intrinsic mechanism of Fourier transform based construction. In contrast, Construction II mainly uses generating functions (GFs), and Construction III Mellin transform which can be regarded as inducing “multiplication-convolution equivalence”, in contrast to Fourier transform inducing “translation-convolution equivalence”. As negative results, we show that the Strategy is not implementable for Inverse Gaussian family (whose densities have a very special structure) and Binomial family (which is discrete but with a finite support); see [Section 5](#).

We provide upper bounds on the variances of the proportion estimators, show their uniform consistency, and provide their speeds of convergence for consistency under independence. Additionally, we have found the following. For estimators given by Construction I, uniform consistency in frequency domain (see [Definition 3](#)) can also be achieved due to the global Lipschitz property of the construction, and their uniform consistency classes (see [Definition 3](#)) can be ordered via set inclusion according to the magnitudes of the moduli of the corresponding CFs; see [Theorem 3](#) and [Corollary 4](#). In contrast, for estimators given by Constructions II and III, uniform consistency in frequency domain is very hard to obtain since the constructions are not Lipschitz transforms of the involved random variables.

For an estimator given by Construction I or Construction II where GFs have finite radii of convergence, its speed of convergence and uniform consistency class do not depend on the supremum norm of the parameter vector (see [Theorem 3](#), [Corollary 4](#) and [Corollary 5](#)), enabling the estimator to be fully data-adaptive. In contrast, for estimators given by other instances of Construction II and Construction III, their speeds of convergence and uniform consistency classes may depend explicitly on the supremum norm of the parameter vector or the infimum of

a transform of this vector (see [Theorem 7](#) and [Theorem 9](#)). However, since the Strategy approximates indicator functions of the status of individual null hypotheses, these speeds and classes always depend on the minimal magnitude of the differences between parameters of interest and their reference value regardless of which among the three constructions is used — a universality phenomenon for the Strategy.

1.3 Notations and conventions

Throughout the article, we use the following conventions and notations: C denotes a generic, positive constant whose values may differ at different occurrences; $O(\cdot)$ and $o(\cdot)$ are respectively Landau’s big O and small o notations; \mathbb{E} and \mathbb{V} are respectively the expectation and variance with respect to the probability measure \Pr ; \mathbb{R} and \mathbb{C} are respectively the set of real and complex numbers; \Re , \Im and \arg denote the real, imaginary part and argument of a complex number, respectively; \mathbb{N} denotes the set of non-negative integers, and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$; δ_y is the Dirac mass at $y \in \mathbb{R}$; ν the Lebesgue measure, and when it is clear that an integral is with respect to ν , the usual notation $d\cdot$ for differential will be used in place of $\nu(d\cdot)$; for $1 \leq p \leq \infty$ and $A \subseteq \mathbb{R}$, $\|f\|_p = (\int_A |f(x)|^p \nu(dx))^{1/p}$ and $L^p(A) = \{f : \|f\|_p < \infty\}$, for which $\|f\|_\infty$ is the essential supremum of f ; for a set A in \mathbb{R} and a scalar a , 1_A is the indicator of A and $A - a = \{x - a : x \in A\}$; \emptyset is the empty set; for $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the integer part of x ; the extended complex plane $\mathbb{C} \cup \{\infty\}$ is identified with the Riemann sphere so that ∞ corresponds to its north pole; for $x \in \mathbb{R}$, $x \rightarrow \infty$ means $x \rightarrow +\infty$, and $x \rightarrow -\infty$ means $x \rightarrow -\infty$; for positive functions $a'(x)$ and $b'(x)$, $a'(x) \sim b'(x)$ means that $\lim_{x \rightarrow \infty} a'(x) (b'(x))^{-1} = 1$.

1.4 Organization of paper

The rest of the article is organized as follows. In [Section 2](#) we formulate the problem of proportion estimation and state the Strategy for constructing proportion estimators. In [Section 3](#) we develop uniformly consistent proportion estimators when the CDFs of random variables have Riemann-Lebesgue type characteristic functions. In [Section 4](#), we construct proportion estimators when the distributions of random variables form discrete NEFs with infinite supports or form NEFs with separable moment sequences. In [Section 5](#), we provide two families of random variables for which the Strategy cannot be applied. In [Section 6](#) and [Section 7](#) we justify the uniform consistency of the constructed proportion estimators for random variables whose distributions form NEFs with infinite supports or with separable moment sequences, respectively. In [Section 8](#) we conduct a simulation study on the proposed estimators, with comparison to that of [Meinshausen and Rice \(2006\)](#). We end the article with a discussion in [Section 9](#) on several topics worthy of future investigations. The supplementary material contains proofs of all theoretical results, provides evidence that Construction III does not apply to Ressel or Hyperbolic Cosine families, and a discussion on uniform consistency of proportion estimators in frequency domain for Construction II and III.

2 The estimation problem and strategy

In this section, we formulate the estimation problem and generalize Jin's strategy. Let z_i , $1 \leq i \leq m$ be m random variables each with mean or median μ_i , such that, for a fixed value μ_0 for the mean or median and some integer m_0 between 0 and m , $\mu_i = \mu_0$ for each $i = 1, \dots, m_0$ and $\mu_i \neq \mu_0$ for each $i = m_0 + 1, \dots, m$. Consider simultaneously testing the null hypothesis $H_{i0} : \mu_i = \mu_0$ versus the alternative hypothesis $H_{i1} : \mu_i \neq \mu_0$ for $1 \leq i \leq m$. Let $I_{0,m} = \{1 \leq i \leq m : \mu_i = \mu_0\}$ and $I_{1,m} = \{1 \leq i \leq m : \mu_i \neq \mu_0\}$. Then the cardinality of $I_{0,m}$ is m_0 , the proportion of true null hypothesis ("null proportion" for short) is defined as $\pi_{0,m} = m^{-1}m_0$, and the proportion of false null hypotheses ("alternative proportion" for short) $\pi_{1,m} = 1 - \pi_{0,m}$. In other words, $\pi_{0,m}$ is the proportion of random variables that have a prespecified mean or median. Our target is to consistently estimate $\pi_{1,m}$ as $m \rightarrow \infty$ when $\{z_i\}_{i=1}^m$ are independent.

We will adopt the following convention from Jin (2008): the dense regime is represented by $\pi_{1,m}$ such that $\liminf_{m \rightarrow \infty} \pi_{1,m} > 0$, the moderately sparse regime by $\pi_{1,m} = Cm^{-s}$ for $s \in (0, 0.5)$, the critically sparse regime by $\pi_{1,m} = Cm^{-0.5}$, and the very sparse regime by $\pi_{1,m} = Cm^{-s}$ for $s \in (0.5, 1)$, where $C > 0$ is a constant.

2.1 The Strategy for proportion estimation

Let $\mathbf{z} = (z_1, \dots, z_m)^T$ and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)^T$. Denote by $F_{\mu_i}(\cdot)$ the CDF of z_i for $1 \leq i \leq m$ and suppose each F_{μ_i} with $0 \leq i \leq m$ is a member of a set \mathcal{F} of CDFs such that $\mathcal{F} = \{F_\mu : \mu \in U\}$ for some non-empty U in \mathbb{R} . For the rest of the paper, we assume that each F_μ is uniquely determined by μ and that U has a non-empty interior.

We state the Strategy below. For each fixed $\mu \in U$, if we can approximate the indicator function $1_{\{\mu \neq \mu_0\}}$ by a function $\psi(t, \mu; \mu_0)$ with $t \in \mathbb{R}$ satisfying $\lim_{t \rightarrow \infty} \psi(t, \mu_0; \mu_0) = 1$ and $\lim_{t \rightarrow \infty} \psi(t, \mu; \mu_0) = 0$ for $\mu \neq \mu_0$, then the "phase function"

$$\varphi_m(t, \boldsymbol{\mu}) = \frac{1}{m} \sum_{i=1}^m [1 - \psi(t, \mu_i; \mu_0)] \quad (1)$$

satisfies $\lim_{t \rightarrow \infty} \varphi_m(t, \boldsymbol{\mu}) = \pi_{1,m}$ for any fixed m and $\boldsymbol{\mu}$ and provides the "Oracle" $\Lambda_m(\boldsymbol{\mu}) = \lim_{t \rightarrow \infty} \varphi_m(t, \boldsymbol{\mu})$. Further, if we can find a function $K : \mathbb{R}^2 \rightarrow \mathbb{R}$ that does not depend on any $\mu \neq \mu_0$ and satisfies the Lebesgue-Stieltjes integral equation

$$\psi(t, \mu; \mu_0) = \int K(t, x; \mu_0) dF_\mu(x), \quad (2)$$

then the "empirical phase function"

$$\hat{\varphi}_m(t, \mathbf{z}) = \frac{1}{m} \sum_{i=1}^m [1 - K(t, z_i; \mu_0)] \quad (3)$$

satisfies $\mathbb{E}[\hat{\varphi}_m(t, \mathbf{z})] = \varphi_m(t, \boldsymbol{\mu})$ for any fixed m, t and $\boldsymbol{\mu}$. Namely, $\hat{\varphi}_m(t, \mathbf{z})$ is an unbiased estimator of $\varphi_m(t, \boldsymbol{\mu})$. When the difference

$$e_m(t) = |\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\mu})| \quad (4)$$

is suitably small for large t , $\hat{\varphi}_m(t, \mathbf{z})$ will accurately estimate $\pi_{1,m}$. Since $\varphi_m(t, \boldsymbol{\mu}) = \pi_{1,m}$ or $\hat{\varphi}_m(t, \mathbf{z}) = \pi_{1,m}$ rarely happens, $\hat{\varphi}_m(t, \mathbf{z})$ usually employs a monotone increasing sequence $\{t_m\}_{m \geq 1}$ such that $\lim_{m \rightarrow \infty} t_m = \infty$ in order to achieve consistency, i.e., to achieve

$$\Pr \left(\left| \frac{\hat{\varphi}_m(t_m, \mathbf{z})}{\pi_{1,m}} - 1 \right| \rightarrow 0 \right) \rightarrow 1 \quad \text{as } m \rightarrow \infty. \quad (5)$$

So, the intrinsic speed for $\hat{\varphi}_m(t_m, \mathbf{z})$ to achieve consistency is better represented by t_m , and we will use t_m as the “speed of convergence” of $\hat{\varphi}_m(t_m, \mathbf{z})$. Throughout the paper, consistency of a proportion estimator is defined via (5) to accommodate the scenario $\lim_{m \rightarrow \infty} \pi_{1,m} = 0$.

By duality, $\varphi_m^*(t, \boldsymbol{\mu}) = 1 - \varphi_m(t, \boldsymbol{\mu})$ is the oracle for which $\pi_{0,m} = \lim_{t \rightarrow \infty} \varphi_m^*(t, \boldsymbol{\mu})$ for any fixed m and $\boldsymbol{\mu}$, $\hat{\varphi}_m^* = 1 - \hat{\varphi}_m(t, \mathbf{z})$ satisfies $\mathbb{E}[\hat{\varphi}_m^*(t, \mathbf{z})] = \varphi_m^*(t, \boldsymbol{\mu})$ for any fixed m, t and $\boldsymbol{\mu}$, and $\hat{\varphi}_m^*(t, \mathbf{z})$ will accurately estimate $\pi_{0,m}$ when $e_m(t)$ is suitably small for large t . Further, the stochastic oscillations of $\hat{\varphi}_m^*(t, \mathbf{z})$ and $\hat{\varphi}_m(t, \mathbf{z})$ are the same and is quantified by $e_m(t)$.

We remark on the differences between the Strategy and Jin’s Strategy. The latter in our notations sets $\mu_0 = 0$, requires $\psi(t, 0; 0) = 1$ for all t , requires location-shift families and uses Fourier transform to construct K and ψ , deals with distributions whose means are equal to their medians, and intends to have

$$1 \geq \psi(t, \mu; 0) \geq 0 \quad \text{for all } \mu \text{ and } t. \quad (6)$$

It is not hard to see from the proof of Lemma 7.1 of Jin (2008) that (6) may not be achievable for non-location-shift families. Further, Jin’s construction of ψ is a special case of solving (2). Finally, it is easier to solve (2) for $K : \mathbb{R}^2 \rightarrow \mathbb{C}$ in the complex domain. However, a real-valued K is preferred for applications in statistics.

3 Construction I

In this section, we present the construction, referred to as “Construction I”, when the set of characteristic functions (CFs) of the CDFs are of Riemann-Lebesgue type (see Definition 1). This construction essentially depends on generalizations of the Riemann-Lebesgue Lemma (“RL Lemma”) and subsumes that by Jin (2008) for Gaussian family.

Recall the family of CDFs $\mathcal{F} = \{F_\mu : \mu \in U\}$ and let $\hat{F}_\mu(t) = \int e^{tx} dF_\mu(x)$ be the CF of F_μ where $\iota = \sqrt{-1}$. Let r_μ be the modulus of \hat{F}_μ . Then $\hat{F}_\mu = r_\mu e^{ih_\mu}$, where $h_\mu : \mathbb{R} \rightarrow \mathbb{R}$ is the multi-valued phase function whose branches differ from each other by addition by a constant

multiple of 2π . However, the multiplicity of h_μ on \mathbb{R} does not affect our analysis since \hat{F}_μ is well-defined on \mathbb{R} . Further, h_μ restricted to the non-empty open interval $(-\tau_\mu, \tau_\mu)$ with $h_\mu(0) = 0$ is uniquely defined, continuous and odd, where $\tau_\mu = \inf \left\{ t > 0 : \hat{F}_\mu(t) = 0 \right\} > 0$; see [Luo and Zhang \(2004\)](#) for a positive lower bound for τ_μ . Note that \hat{F}_μ has no real zeros if and only if $\tau_\mu = \infty$.

Definition 1 *If*

$$\left\{ t \in \mathbb{R} : \hat{F}_{\mu_0}(t) = 0 \right\} = \emptyset \quad (7)$$

and, for each $\mu \in U \setminus \{\mu_0\}$

$$\sup_{t \in \mathbb{R}} \frac{r_\mu(t)}{r_{\mu_0}(t)} < \infty \quad (8)$$

and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_{[-t, t]} \frac{\hat{F}_\mu(y)}{\hat{F}_{\mu_0}(y)} dy = 0, \quad (9)$$

then $\hat{\mathcal{F}} = \left\{ \hat{F}_\mu : \mu \in U \right\}$ is said to be of “Riemann-Lebesgue type (RL type)” (at μ_0 on U).

In [Definition 1](#), condition (7) requires that \hat{F}_{μ_0} have no real zeros, (8) that r_μ with $\mu \in U \setminus \{\mu_0\}$ approximately be of the same order as r_{μ_0} , and (9) forces the “mean value” of $\hat{F}_\mu \hat{F}_{\mu_0}^{-1}$ to converge to zero. Condition (7) excludes the Pólya-type CF $\hat{F}(t) = (1 - |t|) 1_{\{|t| \leq 1\}}$ but holds for an infinitely divisible CF, (8) usually cannot be relaxed to be $r_\mu r_{\mu_0}^{-1} \in L^1(\mathbb{R})$ as seen from [Lemma 1](#) for location-shift families, and (9) induces a generalization of the RL Lemma as given by [Costin et al. \(2016\)](#) (which enables a construction via Fourier transform).

With a Fourier transform based construction comes the subtle issue of determining an “averaging function” ω that helps invoke the RL Lemma and facilitates easy numerical implementation of the resulting proportion estimator. Indeed, a carefully chosen ω will greatly simplify the construction and induce agreeable proportion estimators. We adapt from [Jin \(2008\)](#) the concept of a “good” ω :

Definition 2 *If a function $\omega : [-1, 1] \rightarrow \mathbb{R}$ is non-negative and bounded such that $\int_{[-1, 1]} \omega(s) ds = 1$, then it is called “admissible”. If additionally ω is even on $[-1, 1]$ and continuous on $(-1, 1)$, then it is called “eligible”. If ω is eligible and $\omega(t) \leq \tilde{\omega}(1 - t)$ for all $t \in (0, 1)$ for some convex, super-additive function $\tilde{\omega}$ over $(0, 1)$, then it is called “good”.*

The definition above includes the end points $\{-1, 1\}$ of the compact interval $[-1, 1]$ to tame ω at these points. For example, the triangular density $\omega(s) = \max\{1 - |s|, 0\}$ is good and its CF is

$$\hat{\omega}(t) = \frac{2(1 - \cos(t))}{t^2} 1_{\{t \neq 0\}} + 1_{\{t=0\}}$$

as discussed by [Jin \(2008\)](#). Since $r_\mu(t)$ is even and uniformly continuous in t , an even ω matches r_μ and can induce an even K as a function of t . Unless otherwise noted, ω in this work is always admissible.

Theorem 1 Let $\hat{\mathcal{F}}$ be of RL type and define $K : \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$K(t, x; \mu_0) = \int_{[-1,1]} \frac{\omega(s) \cos(tsx - h_{\mu_0}(ts))}{r_{\mu_0}(ts)} ds. \quad (10)$$

Then $\psi(t, \mu; \mu_0)$ in (2) satisfies the following:

1. $\psi : \mathbb{R} \times U \rightarrow \mathbb{R}$ with

$$\psi(t, \mu; \mu_0) = \int_{[-1,1]} \omega(s) \frac{r_{\mu}(ts)}{r_{\mu_0}(ts)} \cos(h_{\mu}(ts) - h_{\mu_0}(ts)) ds. \quad (11)$$

2. $\psi(t, \mu_0; \mu_0) = 1$ for all t , and $\lim_{t \rightarrow \infty} \psi(t, \mu; \mu_0) = 0$ for each $\mu \in U$ such that $\mu \neq \mu_0$.

For random variables with RL type CFs in general, it is hard to ensure (6), i.e., $1 \geq \psi(t, \mu; 0) \geq 0$ for all μ and t , since we do not have sufficient information on the phase h_{μ} . However, for certain location-shift families, (6) holds when ω is good; see [Corollary 2](#). Under slightly stronger conditions, we have:

Corollary 1 Assume that (7) and (8) hold. If $\frac{r_{\mu}}{r_{\mu_0}} \in L^1(\mathbb{R})$, then the conclusions of [Theorem 1](#) hold.

When each F_{μ} has a density with respect to ν , the condition $r_{\mu} r_{\mu_0}^{-1} \in L^1(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ in [Corollary 1](#) can be fairly strong since it forces $\lim_{t \rightarrow \infty} \frac{r_{\mu}(t)}{r_{\mu_0}(t)} = 0$. Unfortunately, [Corollary 1](#) is not applicable to location-shift families since for these families $\frac{r_{\mu}(t)}{r_{\mu_0}(t)} \equiv 1$ for all $\mu \in U$ whenever $r_{\mu_0}(t) \neq 0$; see [Lemma 1](#). In contrast, [Theorem 1](#) is as we explain next.

Recall the definition of location-shift family, i.e., $\mathcal{F} = \{F_{\mu} : \mu \in U\}$ is a location-shift family if and only if $z + \mu'$ has CDF $F_{\mu+\mu'}$ whenever z has CDF F_{μ} for $\mu, \mu + \mu' \in U$.

Lemma 1 Suppose \mathcal{F} is a location-shift family. Then, r_{μ} does not depend on μ and $h_{\mu}(t) = t\mu'$, where $\mu' = \mu - \mu_0$. If in addition $\hat{F}_{\mu_0}(t) \neq 0$ for all $t \in \mathbb{R}$, then $\hat{\mathcal{F}}$ is of RL type.

[Lemma 1](#) implies that very likely the set of location-shift family distributions is a subset of the set of distributions with RL type CFs. However, verifying if \hat{F}_{μ_0} for a location-shift family has any real zeros is quite difficult in general. Nonetheless, with [Lemma 1](#), [Theorem 1](#) implies:

Corollary 2 If \mathcal{F} is a location-shift family for which (7) holds, then

$$\psi(t, \mu; \mu_0) = \int K(t, y + (\mu - \mu_0); \mu_0) dF_{\mu_0}(y) = \int_{[-1,1]} \omega(s) \cos(ts(\mu - \mu_0)) ds. \quad (12)$$

If additionally ω is good, then (6) holds, i.e., $1 \geq \psi(t, \mu; \mu_0) \geq 0$ for all μ and t .

The identity (12) in [Corollary 2](#) asserts that, for a location-shift family, ψ in (2) reduces to a “convolution”, and it is the intrinsic mechanism behind the Fourier transform based construction. When $\mu_0 = 0$, from [Corollary 2](#) we directly have

$$\psi(t, \mu; 0) = \int K(t, x + \mu; 0) dF_0(x) = \int_{[-1,1]} \omega(s) \cos(ts\mu) ds \quad (13)$$

recovering the construction by [Jin \(2008\)](#) for Gaussian and Laplace families. Examples for which (13) holds are given in [Section 3.1](#).

3.1 Construction I: some examples

We provide some examples from Construction I. They are all formed out of location-shift families by allowing the location parameter μ to vary but fixing the scale parameter σ , and are infinitely divisible so that none of their CFs has any real zero; see [Pitman and Yor \(2003\)](#), [Lukacs \(1970\)](#) and [Fischer \(2014\)](#) for details on this. Note however that the Poisson family as an example given in [Section 4.3](#) is infinitely divisible but does not have RL type CFs. There are certainly other examples that Construction I applies to. But we will not attempt to exhaust them.

Example 1 *Gaussian family* Normal (μ, σ^2) with mean μ and standard deviation $\sigma > 0$, for which

$$\frac{dF_\mu}{d\nu}(x) = f_\mu(x) = \left(\sqrt{2\pi}\sigma\right)^{-1} \exp\left(-2^{-1}\sigma^{-2}(x - \mu)^2\right).$$

The CF of f_μ is $\hat{f}_\mu(t) = \exp(it\mu) \exp(-2^{-1}t^2\sigma^2)$. Here $r_\mu^{-1}(t) = \exp(2^{-1}t^2\sigma^2)$. Therefore,

$$K(t, x; \mu_0) = \int_{[-1,1]} r_\mu^{-1}(ts) \omega(s) \exp(itx(x - \mu_0)) ds \quad (14)$$

and (12) holds. When $\mu_0 = 0$ and $\sigma = 1$, this construction reduces to that in [Section 2.1](#) of [Jin \(2008\)](#).

Example 2 *Laplace family* Laplace $(\mu, 2\sigma^2)$ with mean μ and standard deviation $\sqrt{2}\sigma > 0$ for which

$$\frac{dF_\mu}{d\nu}(x) = f_\mu(x) = \frac{1}{2\sigma} \exp(-\sigma^{-1}|x - \mu|)$$

and the CF is $\hat{f}_\mu(t) = (1 + \sigma^2 t^2)^{-1} \exp(it\mu)$. Therefore, (14) and (12) hold with $r_\mu^{-1}(t) = 1 + \sigma^2 t^2$. When $\mu_0 = 0$ and $\sigma = 1$, this construction reduces to that in [Section 7.1](#) of [Jin \(2008\)](#).

Example 3 *Logistic family* Logistic (μ, σ) with mean μ and scale parameter $\sigma > 0$, for which

$$\frac{dF_\mu}{d\nu}(x) = f_\mu(x) = \frac{1}{4\sigma} \operatorname{sech}^2\left(\frac{x - \mu}{2\sigma}\right)$$

and the CF of f_μ is $\hat{f}_\mu(t) = \pi\sigma t (\sinh(\pi\sigma t))^{-1} \exp(t\mu)$. Therefore, (14) and (12) hold with $r_\mu^{-1}(t) = (\pi\sigma t)^{-1} \sinh(\pi\sigma t) \sim (2\pi\sigma t)^{-1} e^{\pi\sigma t}$ as $t \rightarrow \infty$.

Example 4 Cauchy family $\text{Cauchy}(\mu, \sigma)$ with median μ and scale parameter $\sigma > 0$, for which

$$\frac{dF_\mu}{d\nu}(x) = f_\mu(x) = \frac{1}{\pi\sigma} \frac{\sigma^2}{(x - \mu)^2 + \sigma^2}$$

and the CF is $\hat{f}_\mu(t) = \exp(-\sigma|t|) \exp(t\mu)$. Therefore, (14) and (12) hold with $r_\mu^{-1}(t) = \exp(\sigma|t|)$. Note that the Cauchy family does not have first-order absolute moment.

Example 5 The Hyperbolic Secant family $\text{HSecant}(\mu, \sigma)$ with mean μ and scale parameter $\sigma > 0$, for which

$$\frac{dF_\mu}{d\nu}(x) = f_\mu(x) = \frac{1}{2\sigma} \frac{1}{\cosh\left(\pi \frac{x-\mu}{\sigma}\right)};$$

see, e.g., Chapter 1 of [Fischer \(2014\)](#). The identity

$$\int_{-\infty}^{+\infty} e^{tx} \frac{dx}{\pi \cosh(x)} = \cosh(2^{-1}\pi t),$$

implies $\hat{F}_\mu(t) = \sigma^{-1} \exp(-t\mu\sigma^{-1}) \text{sech}(t\sigma^{-1})$. Therefore, (14) and (12) hold with $r_\mu^{-1}(t) = \sigma \cosh(t\sigma^{-1}) \sim 2^{-1}\sigma \exp(\sigma^{-1}t)$ as $t \rightarrow \infty$.

3.2 Construction I: uniform consistency and speed of convergence

The performance of the estimator $\hat{\varphi}_m$ depends on how accurately it approximates φ_m , the oracle that knows the true value $\pi_{1,m}$ as $t \rightarrow \infty$. Specifically, the smaller $e_m(t)$ defined by (4) is when t is large, the more accurately $\hat{\varphi}_m(t; \mathbf{z})$ estimates $\pi_{1,m}$. Two key factors that affect e_m are: (i) the magnitude of the reciprocal of the modulus, $r_{\mu_0}^{-1}(t)$, which appears as a scaling factor in the integrand in the definition of K in (10), and (ii) the magnitudes of the μ_i 's and the variabilities of z_i 's. As will be shown by [Theorem 2](#), for independent $\{z_i\}_{i=1}^m$, the oscillation of $e_m(t)$ depends mainly on $r_{\mu_0}^{-1}(t)$ due to concentration of measure for independent, uniformly bounded random variables and their transforms by Lipschitz functions, whereas the consistency of $\hat{\varphi}_m(t; \mathbf{z})$ depends also on the magnitudes of $\pi_{1,m}$ and μ_i 's that affect how accurate the Oracle is when t is large.

Theorem 2 If $\hat{\mathcal{F}}$ is of RL type and $\{z_i\}_{i=1}^m$ are independent. Let $a(t; \mu_0) = \int_{[-1,1]} \frac{ds}{r_{\mu_0}(ts)}$ for $t \in \mathbb{R}$. Then

$$\mathbb{V} [|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\mu})|] \leq \frac{1}{m} \|\omega\|_\infty^2 a^2(t; \mu_0). \quad (15)$$

Further, for any fixed $\lambda > 0$, with probability at least $1 - 2 \exp(-2^{-1}\lambda^2)$,

$$|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\mu})| \leq \frac{\lambda \|\omega\|_\infty a(t; \mu_0)}{\sqrt{m}}. \quad (16)$$

If there are positive sequences $\{u_m\}_{m \geq 1}$, $\{\lambda_m\}_{m \geq 1}$ and $\{t_m\}_{m \geq 1}$ such that

$$\lim_{m \rightarrow \infty} \sup \{ \psi(t, \mu; \mu_0) : (t, |\mu|) \in [t_m, \infty) \times [u_m, \infty) \} = 0 \quad (17)$$

and

$$\lim_{m \rightarrow \infty} \frac{\lambda_m a(t_m; \mu_0)}{\pi_{1,m} \sqrt{m}} = 0 \quad \text{and} \quad \lim_{m \rightarrow \infty} \exp(-2^{-1} \lambda_m^2) = 0, \quad (18)$$

then

$$\Pr \left(\left| \pi_{1,m}^{-1} \hat{\varphi}_m(t_m, \mathbf{z}) - 1 \right| \rightarrow 0 \right) \rightarrow 1 \quad (19)$$

whenever $\{\mu_i : i \in I_{1,m}\} \subseteq [u_m, \infty)$.

Theorem 2 bounds the variance of $\hat{\varphi}_m(t, \mathbf{z})$, captures the key ingredients needed for and the essence of proving the consistency of a proportion estimator based on Construction I, and shows that such a proportion estimator is consistent as long as $\lambda_m \rightarrow \infty$, $\lambda_m \pi_{1,m}^{-1} a(t_m; \mu_0)$ is of smaller order than \sqrt{m} and each $\psi(t_m, \mu_i; \mu_0)$ with $i \in I_{1,m}$ is negligible for a sequence $t_m \rightarrow \infty$. Using **Theorem 2**, we can characterize the consistency of the estimator $\hat{\varphi}_m(t, \mathbf{z})$ for each example given in **Section 3.1** as follows:

Corollary 3 Consider $\hat{\varphi}(t_m, \mathbf{z})$ from Construction I. Let $\{z_i\}_{i=1}^m$ be independent and $\{t_m : m \geq 1\}$ a positive sequence such that $\lim_{m \rightarrow \infty} t_m = \infty$. Assume $\min_{i \in I_{1,m}} |\mu_i - \mu_0| \geq (t_m)^{-1} \log \log m$. Then $\Pr \left(\left| \pi_{1,m}^{-1} \hat{\varphi}_m(t_m, \mathbf{z}) - 1 \right| \rightarrow 0 \right) \rightarrow 1$ holds

- for Gaussian, Hyperbolic Secant, Logistic and Cauchy family respectively when $t_m = \sigma^{-1} \sqrt{2\gamma \log m}$, $t_m = \sigma \gamma \log m$, $t_m = (\sigma \pi)^{-1} \gamma \log m$ and $t_m = \sigma^{-1} \gamma \log m$, $\lambda_m = o(t_m)$, $\lambda_m \rightarrow \infty$ and $\pi_{1,m} \geq C m^{\gamma-0.5}$ with $\gamma \in (0, 0.5]$;
- for Laplace family when $t_m = \log m$, $\lambda_m = O(t_m)$, $\lambda_m \rightarrow \infty$ and $\pi_{1,m} \geq C m^{-\gamma}$ with $\gamma \in [0, 0.5)$,

where $C > 0$ can be any constant for which $\pi_{1,m} \in (0, 1]$ as γ varies in its designated range respectively.

To characterize if an estimator $\hat{\varphi}_m(t, \mathbf{z})$ is uniformly consistent with respect to $\pi_{1,m}$ and t , we introduce the following definition:

Definition 3 Given a family \mathcal{F} , the sequence of sets $\mathcal{Q}_m(\boldsymbol{\mu}, t; \mathcal{F}) \subseteq \mathbb{R}^m \times \mathbb{R}$ for each $m \in \mathbb{N}_+$ is called a “uniform consistency class” for the estimator $\hat{\varphi}_m(t, \mathbf{z})$ if

$$\Pr \left(\sup_{\boldsymbol{\mu} \in \mathcal{Q}_m(\boldsymbol{\mu}, t; \mathcal{F})} \left| \pi_{1,m}^{-1} \sup_{t \in \mathcal{Q}_m(\boldsymbol{\mu}, t; \mathcal{F})} \hat{\varphi}_m(t, \mathbf{z}) - 1 \right| \rightarrow 0 \right) \rightarrow 1. \quad (20)$$

If (20) holds and each $\mathcal{Q}_m(\boldsymbol{\mu}, t; \mathcal{F})$ contains an open subset $P_m \subseteq [0, 1]$ that contains $\pi_{1,m}$ and $\liminf_{m \rightarrow \infty} P_m \neq \emptyset$, then $\hat{\varphi}_m(t, \mathbf{z})$ is said to be “uniformly consistent”. If further each

$\mathcal{Q}_m(\boldsymbol{\mu}, t; \mathcal{F})$ contains a connected subset $G_m \subseteq \mathbb{R}$ such that $\lim_{m \rightarrow \infty} \nu(G_m) = \infty$, then $\hat{\varphi}_m(t, \mathbf{z})$ is said to be “uniformly consistent in frequency domain”.

Now we discuss uniform consistency in frequency domain of an estimator from Construction I. Define

$$\mathcal{B}_m(\rho) = \left\{ \boldsymbol{\mu} \in \mathbb{R}^m : m^{-1} \sum_{i=1}^m |\mu_i - \mu_0| \leq \rho \right\} \quad \text{for some } \rho > 0$$

and $u_m = \min \{ |\mu_j - \mu_0| : \mu_j \neq \mu_0 \}$.

Theorem 3 Assume that \mathcal{F} is a location-shift family for which (7) holds and $\int |x|^2 dF_\mu(x) < \infty$ for each $\mu \in U$. If $\sup_{y \in \mathbb{R}} \left| \frac{d}{dy} h_{\mu_0}(y) \right| = C_{\mu_0} < \infty$, then for the estimator $\hat{\varphi}_m(t, \mathbf{z})$ from Construction I, a uniform consistency class is

$$\mathcal{Q}_m(\boldsymbol{\mu}, t; \mathcal{F}) = \left\{ \begin{array}{l} q\gamma' > \vartheta > 2^{-1}, \gamma' > 0, \gamma'' > 0, 0 \leq \vartheta' < \vartheta - 1/2, \\ R_m(\rho) = O\left(m^{\vartheta'}\right), \tau_m \leq \gamma_m, u_m \geq \frac{\log \log m}{\gamma'' \tau_m}, \\ t \in [0, \tau_m], \lim_{m \rightarrow \infty} \pi_{1,m}^{-1} \Upsilon(q, \tau_m, \gamma_m, r_{\mu_0}) = 0 \end{array} \right\} \quad (21)$$

where $q, \gamma', \gamma'', \vartheta$ and ϑ' are constants, $R_m(\rho) = \int |x| dF_{\mu_0}(x) + 2\rho + 2C_{\mu_0}$, $\gamma_m = \gamma' \log m$ and

$$\Upsilon(q, \tau_m, \gamma_m, r_{\mu_0}) = \frac{2 \|\omega\|_\infty \sqrt{2q\gamma_m}}{\sqrt{m}} \sup_{t \in [0, \tau_m]} \int_{[0,1]} \frac{ds}{r_{\mu_0}(ts)}. \quad (22)$$

Moreover, for all sufficiently large m ,

$$\sup_{\boldsymbol{\mu} \in \mathcal{B}_m(\rho)} \sup_{t \in [0, \tau_m]} |\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\mu})| \leq \Upsilon(q, \tau_m, \gamma_m, r_{\mu_0}) \quad (23)$$

holds with probability at least $1 - o(1)$.

Several remarks on [Theorem 3](#) are ready to be stated. Firstly, since [Theorem 3](#) requires the random variables to have finite absolute second-order moments, it may not apply to location-shift families that do not have first-order absolute moments. Secondly, even though [Theorem 3](#) potentially allows for many possible choices of t for $\hat{\varphi}_m(t, \mathbf{z})$, we should choose τ_m such that $\tau_m \rightarrow \infty$ as fast as possible so that $\pi_{1,m}^{-1} \hat{\varphi}_m(\tau_m, \mathbf{z}) \rightarrow 1$ as fast as possible. Thirdly, compared to [Theorems 1.4 and 1.5 of Jin \(2008\)](#) where $\mathcal{B}_m(\rho)$ is for a fixed ρ for Gaussian family, we allow $\rho \rightarrow \infty$ for location-shift families. Fourthly, the bound (23) together with (22) imply that, when other things are kept fixed, the larger r_μ^{-1} is, the slower $\pi_{1,m}^{-1} \hat{\varphi}_m(\tau_m, \mathbf{z}) \rightarrow 1$. This has been observed by [Jin \(2008\)](#) for the Gaussian and Laplace families since $r_\mu^{-1}(t)$ for the former is much larger than the latter when t is large. Fifthly, for location-shift families the constants q, γ', γ'' and ϑ can be specified by a user. If additionally $\vartheta' = 0$, then $\mathcal{Q}_m(\boldsymbol{\mu}, t; \mathcal{F})$ is fully data-adaptive and depends only on $u_m = \min_{i \in I_{1,m}} |\mu_i - \mu_0|$; see also [Corollary 4](#) on this for examples given in [Section 3.1](#).

Let \mathcal{F} and $\tilde{\mathcal{F}}$ be two location-shift families that have RL type CFs and are determined by the same set of parameters, and r_{μ_0} and \tilde{r}_{μ_0} respectively the moduli of the CFs of $F_{\mu_0} \in \mathcal{F}$ and $\tilde{F}_{\mu_0} \in \tilde{\mathcal{F}}$. For the estimator $\hat{\varphi}_m(t, \mathbf{z})$ under the same settings and under the conditions stated in [Theorem 3](#), consider the two uniform consistency classes $\mathcal{Q}_m(\boldsymbol{\mu}, t; \mathcal{F})$ and $\mathcal{Q}_m(\boldsymbol{\mu}, t; \tilde{\mathcal{F}})$ for $\hat{\varphi}_m(t, \mathbf{z})$ that have the same constants q, γ', ϑ and γ'' . Then [Theorem 3](#) implies that

$$r_{\mu_0}(t) \leq \tilde{r}_{\mu_0}(t) \text{ for all large } t > 0 \implies \mathcal{Q}_m(\boldsymbol{\mu}, t; \mathcal{F}) \supseteq \mathcal{Q}_m(\boldsymbol{\mu}, t; \tilde{\mathcal{F}}) \quad (24)$$

In other words, for location-shift families that have RL type CFs, the uniform consistency classes for estimators from Construction I can be reversely ordered via set inclusion by the magnitudes of the associated moduli when other things are kept fixed.

Another consequence of [Theorem 3](#) is as follows. If we fix $q, \gamma', \gamma'', \vartheta$ and ϑ' , then r_{μ_0} determines $\mathcal{Q}_m(\boldsymbol{\mu}, t; \mathcal{F})$. In particular, if F_{μ} has a density with respect to ν , then $\lim_{t \rightarrow \infty} r_{\mu}(t) = 0$ must hold, which forces (22) to give

$$\Upsilon(q, \tau_m, \gamma' \log m, r_{\mu_0}) \geq \frac{C\sqrt{q\gamma' \log m}}{\sqrt{m}} \text{ as } m \rightarrow \infty. \quad (25)$$

Since Hoeffding inequality, used in the proof of [Theorem 3](#), is optimal for independent, uniformly almost surely bounded random variables induced by Construction I, [Theorem 3](#) and (25) together imply that, when Construction I applied to location-shift families with absolutely continuous CDFs, a uniform consistency class for the corresponding proportion estimator is unlikely able to contain any $\pi_{1,m} \in (0, m^{-0.5}]$. Comparing this with the conclusions of [Theorem 2](#) and [Corollary 3](#), we see that a sacrifice to achieve uniform consistency in frequency domain is the reduction of the range for $\pi_{1,m}$ for which $\hat{\varphi}_m(t, \mathbf{z})$ can be consistent.

Finally, [Theorem 3](#) gives

Corollary 4 *When $\{z_i\}_{i=1}^m$ are independent, the following hold for $\hat{\varphi}_m(t, \mathbf{z})$:*

1. *For Gaussian family: $q\sigma^{-1} > \vartheta > 2^{-1}, 0 \leq \vartheta' < \vartheta - 1/2, \gamma_m = \sigma^{-1} \log m$ and*

$$\mathcal{Q}_m(\boldsymbol{\mu}, t; \mathcal{F}) = \left\{ \begin{array}{l} R_m(\rho) = O\left(m^{\vartheta'}\right), u_m \geq \frac{\log \log m}{\sqrt{2 \log m}}, 0 < \gamma \leq 0.5 \\ \pi_{1,m} \geq Cm^{\gamma-0.5}, t \in [0, \sigma^{-1} \sqrt{2\gamma \log m}] \end{array} \right\} \quad (26)$$

The fastest speed of convergence is $\sqrt{\log m}$, achieved when $\liminf_{m \rightarrow \infty} \pi_{1,m} > 0$.

2. *For Laplace family: $q > \vartheta > 2^{-1}, 0 \leq \vartheta' < \vartheta - 1/2, \gamma_m = \log m$ and*

$$\mathcal{Q}_m(\boldsymbol{\mu}, t; \mathcal{F}) = \left\{ \begin{array}{l} R_m(\rho) = O\left(m^{\vartheta'}\right), u_m \geq \frac{\log \log m}{\log m}, 0 \leq \gamma < 1/2 \\ \pi_{1,m} \geq Cm^{-\gamma}, t \in [0, \log m] \end{array} \right\}.$$

3. *Hyperbolic Secant family*: $q\sigma > \vartheta > 2^{-1}$, $\gamma_m = \sigma \log m$, $0 \leq \vartheta' < \vartheta - 1/2$ and

$$\mathcal{Q}_m(\boldsymbol{\mu}, t; \mathcal{F}) = \left\{ \begin{array}{l} R_m(\rho) = O\left(m^{\vartheta'}\right), u_m \geq \frac{\log \log m}{\log m}, 0 < \gamma < 1/2 \\ \pi_{1,m} \geq Cm^{\gamma-0.5}, t \in [0, \sigma\gamma \log m] \end{array} \right\}.$$

4. *Logistic family*: $q(\sigma\pi)^{-1} > \vartheta > 2^{-1}$, $\gamma_m = (\sigma\pi)^{-1} \log m$, $0 \leq \vartheta' < \vartheta - 1/2$ and

$$\mathcal{Q}_m(\boldsymbol{\mu}, t; \mathcal{F}) = \left\{ \begin{array}{l} R_m(\rho) = O\left(m^{\vartheta'}\right), u_m \geq \frac{\log \log m}{\log m}, 0 < \gamma < 1/2 \\ \pi_{1,m} \geq Cm^{\gamma-0.5}, t \in [0, (\sigma\pi)^{-1} \gamma \log m] \end{array} \right\}.$$

In each case above, $C > 0$ can be any constant for which $\pi_{1,m} \in (0, 1]$ as γ varies in its designated range.

Corollary 4 provides uniform consistency classes for estimators from Construction I when it is applied to five location-shift families with general scale parameter $\sigma > 0$. In particular, if $\sigma = 1$ and $q > 3/2$, $\vartheta = q/3$ and $\vartheta' = 0$ is set in (26), then we recover the uniform consistency class given by Theorems 1.4 and 1.5 of Jin (2008).

4 Construction II and Construction III

When the CDFs of $\{z_i\}_{i=1}^m$ do not have RL type CFs, Construction I in Section 3 cannot be used. In particular, outside location-shift families, the translation-convolution equivalence does not hold and Hoeffding inequality is no longer applicable. This makes the construction of a proportion estimator using the Strategy much more challenging. So, we will restrict our attention to $\{z_i\}_{i=1}^m$ whose CDFs do not have RL type CFs but belong to NEFs whose mean and variance are functionally related. Specifically, we show that the Strategy is implementable for discrete NEFs with infinite supports or “separable moment functions” (see Definition 4). These include 8 of the total of 12 NEF-CVFs. The techniques of construction mainly use generating functions (GFs) and Mellin transform.

4.1 A brief review on natural exponential families

We provide a very brief review on NEF, whose details can be found in Letac (1992). Let β be a positive Radon measure on \mathbb{R} that is not concentrated on one point. Let $L(\theta) = \int e^{x\theta} \beta(dx)$ for $\theta \in \mathbb{R}$ be its Laplace transform and Θ be the maximal open set containing θ such that $L(\theta) < \infty$. Suppose Θ is not empty and let $\kappa(\theta) = \log L(\theta)$ be the cumulant function of β . Then

$$\mathcal{F} = \{G_\theta : G_\theta(dx) = \exp(\theta x - \kappa(\theta)) \beta(dx), \theta \in \Theta\}$$

forms an NEF with respect to the basis β . Note that Θ has a non-empty interior if it is not empty and that L is analytic on the strip $A_\Theta = \{z \in \mathbb{C} : \Re(z) \in \Theta\}$.

The NEF \mathcal{F} can be equivalently characterized by its mean domain and variance function. Specifically, the mean function $\mu : \Theta \rightarrow U$ with $U = \mu(\Theta)$ is given by $\mu(\theta) = \frac{d}{d\theta}\kappa(\theta)$, and the variance function is $V(\theta) = \frac{d^2}{d\theta^2}\kappa(\theta)$ and can be parametrized by μ as

$$V(\mu) = \int (x - \mu)^2 F_\mu(dx) \text{ for } \mu \in U,$$

where $\theta = \theta(\mu)$ is the inverse function of μ and $F_\mu = G_{\theta(\mu)}$. Namely, $\mathcal{F} = \{F_\mu : \mu \in U\}$. The pair (V, U) is called the variance function of \mathcal{F} , and it characterizes \mathcal{F} .

For the constructions of proportion estimators for NEFs, we will reuse the notation K but take K as a function of t and $\theta \in \Theta$. Note that K depends on θ_0 but not on any $\theta \neq \theta_0$. Further, we will reuse the notation ψ but take it as a function of t and $\theta \in \Theta$. For an NEF, ψ defined by (2) becomes

$$\psi(t, \theta; \theta_0) = \int K(t, x; \theta_0) dG_\theta(x) \text{ for } G_\theta \in \mathcal{F}.$$

Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_m)$. Then accordingly $\hat{\varphi}_m(t, \mathbf{z}) = \frac{1}{m} \sum_{i=1}^m [1 - K(t, z_i; \theta_0)]$ and $\varphi_m(t, \boldsymbol{\theta}) = \frac{1}{m} \sum_{i=1}^m [1 - \psi(t, \theta_i; \theta_0)]$.

4.2 Construction II: discrete NEFs with infinite supports

Suppose the basis β for \mathcal{F} is discrete with support \mathbb{N} , i.e., there exists a positive sequence $\{c_k\}_{k \geq 0}$ such that

$$\beta = \sum_{k=0}^{\infty} c_k \delta_k. \quad (27)$$

Then the power series $H(z) = \sum_{k=0}^{\infty} c_k z^k$ with $z \in \mathbb{C}$ must have a positive radius of convergence R_H , and h is the generating function (GF) of β . Further, if β is a probability measure, then $(-\infty, 0] \subseteq \Theta$ and $R_H \geq 1$, and vice versa. The following approach, which we refer to as ‘‘Construction II’’, provides the construction for discrete NEFs with support \mathbb{N} .

Theorem 4 *Let \mathcal{F} be the NEF generated by β in (27) and ω admissible. For $x \in \mathbb{N}$ and $t \in \mathbb{R}$ set*

$$K(t, x; \theta_0) = H(e^{\theta_0}) \int_{[-1,1]} \frac{(ts)^x \cos\left(\frac{\pi x}{2} - tse^{\theta_0}\right)}{H^{(x)}(0)} \omega(s) ds. \quad (28)$$

Then

$$\psi(t, \theta; \theta_0) = \int K(t, x; \theta_0) dG_\theta(x) = \frac{H(e^{\theta_0})}{H(e^\theta)} \int_{[-1,1]} \cos\left(st(e^\theta - e^{\theta_0})\right) \omega(s) ds,$$

$\psi(t, \theta_0; \theta_0) = 1$ for any t , and $\lim_{t \rightarrow \infty} \psi(t, \theta; \theta_0) = 0$ for each $\theta \neq \theta_0$.

In **Theorem 4**, $H^{(k)}(0) = c_k k!$ for $k \in \mathbb{N}$. So, for Construction II, if β is known, then we can use $c_k k!$ instead of $H^{(k)}(0)$, whereas if H is known and $H^{(k)}$ is easy to compute, we can use $H^{(k)}(0)$. This will greatly aid the numerical implementation of Construction II.

4.3 Construction II: some examples

Theorem 4 covers the construction for Abel, Negative Binomial, Poisson, Strict Arcsine, Large Arcsine and Takács families, each of which is an NEF-CVF, has basis β with support \mathbb{N} , is infinite divisible such that $H(e^\theta) \neq 0$ for each $\theta \in \Theta$, and has non-RL type CFs; see **Letac and Mora (1990)** for details on these distributions. However, for each of Abel and Large Arcsine families, the corresponding GF is a composition of two analytic functions, and manually computing $H(e^{\theta_0})$ in the construction of K in the statement of **Theorem 4** may be cumbersome.

Example 6 *Poisson family* Poisson(μ) with mean $\mu > 0$, for which $F_\mu(\{k\}) = \Pr(X = k) = \frac{\mu^k}{k!} e^{-\mu}$ for $k \in \mathbb{N}$. Clearly,

$$\hat{F}_\mu(t) = \exp(\mu(e^{it} - 1)) = \exp(\mu(\cos t - 1)) \exp(i\mu \sin t).$$

However, $\{\hat{F}_\mu : \mu > 0\}$ are not of RL type since when $\mu = \mu_0 + 1$ and $t > 0$,

$$\begin{aligned} \frac{1}{t} \Re \left(\int_{[-t,t]} \frac{\hat{F}_\mu(y)}{\hat{F}_{\mu_0}(y)} dy \right) &= \frac{e^{-1}}{t} \int_{[-t,t]} \exp(\cos y) \cos(\sin y) dy \\ &\geq 2e^{-1} e^{-1} \cos 1 > 0. \end{aligned}$$

The basis is $\beta = \sum_{k=0}^{\infty} \frac{1}{k!} \delta_k$, $L(\theta) = \exp(e^\theta)$ with $\theta \in \mathbb{R}$, $\mu(\theta) = e^\theta$, $H(z) = e^z$ with $R_H = \infty$ and $H^{(k)}(0) = 1$ for all $k \in \mathbb{N}$.

Example 7 *Negative Binomial family* NegBinomial(θ, n) with $\theta < 0$ and $n \in \mathbb{N}_+$ such that

$$G_\theta(\{k\}) = \Pr(X = k) = \frac{c_k^*}{k!} e^{k\theta} (1 - e^\theta)^n$$

with $c_k^* = \frac{(k+n-1)!}{(n-1)!}$ for $k \in \mathbb{N}$. The basis is $\beta = \sum_{k=0}^{\infty} \frac{c_k^*}{k!} \delta_k$, $L(\theta) = (1 - e^\theta)^{-n}$ with $\theta < 0$, $\mu(\theta) = ne^\theta (1 - e^\theta)^{-1}$, $H(z) = (1 - z)^{-n}$ with $R_H = 1$, and $H^{(k)}(0) = c_k^*$ for all $k \in \mathbb{N}$.

Example 8 *Strict Arcsine family*. Its VF is $V(u) = u(1 + u^2)$ and $\beta = \sum_{n=0}^{\infty} \frac{c_n^*(1)}{n!} \delta_n$, where $c_0^*(1) = c_1^*(1) = 1$,

$$c_{2n}^*(\sigma) = \prod_{k=0}^{n-1} (\sigma^2 + 4k^2) \quad \text{and} \quad c_{2n+1}^*(\sigma) = \sigma \prod_{k=0}^{n-1} (\sigma^2 + (2k+1)^2) \quad (29)$$

for $\sigma > 0$ and $n \in \mathbb{N}_+$. Further, $H(z) = \exp(\arcsin z)$ with $R_H = 1$.

Example 9 *Large Arcsine family.* Its VF is $V(u) = u(1 + 2u + 2u^2)$ and $\beta = \sum_{n=0}^{\infty} c_n \delta_n$ with $c_n = \frac{c_n^*(1+n)}{(n+1)!}$ for $n \in \mathbb{N}$, where $c_n^*(\sigma)$ is defined in (29) and for which $H(z) = \exp(\arcsin(h(z)))$ with $h(z) = \sum_{k=0}^{\infty} c_k z^{k+1}$. It can be seen that R_H must be finite; otherwise, $\lim_{|z_l| \rightarrow \infty} |h(z_l)| = \infty$ for a sequence $\{z_l : l \geq 1\}$, and $H(z_l)$ cannot be expanded into a convergent power series at z_l for l sufficiently large.

Example 10 *Abel family.* Its VF $V(u) = u(1 + u)^2$, $\beta = \sum_{k=0}^{\infty} c_k \delta_k$ with $c_k = \frac{(1+k)^{k-1}}{k!}$ for $k \in \mathbb{N}$, and $H(z) = e^{h(z)}$ with $h(z) = \sum_{k=0}^{\infty} c_k z^{k+1}$ with $R_H = e^{-1}$.

Example 11 *Takács family.* Its VF is $V(u) = u(1 + u)(1 + 2u)$ and $\beta = \delta_0 + \sum_{k=1}^{\infty} c_k \delta_k$ with $c_k = \frac{(2k)!}{k!(k+1)!}$ for $k \in \mathbb{N}_+$, for which $H(z) = \frac{1 - \sqrt{1-4z}}{2z}$ with $R_H = 4^{-1}$ and $z = 0$ is a removable singularity of H .

The above calculations show that the GFs of Negative Binomial, Strict Arcsine, Large Arcsine, Abel and Takács families all have positive and finite radii of convergence whereas that of Poisson family has infinite radius of convergence. This will be very helpful in determining uniform consistency classes for Construction II and its numerical implementation; see [Corollary 5](#).

4.4 Construction III: continuous NEFs with separable moments

In contrast to NEFs with support \mathbb{N} , we consider non-location-shift NEFs whose members are continuous distributions. Assume $0 \in \Theta$, so that β is a probability measure with finite moments of all orders. Let

$$\tilde{c}_n(\theta) = \frac{1}{L(\theta)} \int x^n e^{\theta x} \beta(dx) = \int x^n dG_{\theta}(x) \quad \text{for } n \in \mathbb{N} \quad (30)$$

be the moment sequence for $G_{\theta} \in \mathcal{F}$. Note that (30) is the Mellin transform of the measure G_{θ} .

Definition 4 *If there exist two functions $\zeta, \xi : \Theta \rightarrow \mathbb{R}$ and a sequence $\{\tilde{a}_n\}_{n \geq 0}$ that satisfy the following:*

- $\xi(\theta) \neq \xi(\theta_0)$ whenever $\theta \neq \theta_0$, $\zeta(\theta) \neq 0$ for all $\theta \in U$, and ζ does not depend on any $n \in \mathbb{N}$,
- $\tilde{c}_n(\theta) = \xi^n(\theta) \zeta(\theta) \tilde{a}_n$ for each $n \in \mathbb{N}$ and $\theta \in \Theta$,
- $\Psi(t, \theta) = \sum_{n=0}^{\infty} \frac{t^n \xi^n(\theta)}{\tilde{a}_n n!}$ is absolutely convergent pointwise in $(t, \theta) \in \mathbb{R} \times \Theta$,

then the moment sequence $\{\tilde{c}_n(\theta)\}_{n \geq 0}$ is called “separable” (at θ_0).

The concept of separable moment sequence is an analogy to the structured integrand used in (68) for Construction II, and the condition on $\Psi(t, \theta)$ is usually satisfied since $\sum_{n=0}^{\infty} \frac{t^n \xi^n(\theta)}{n!}$ already is convergent pointwise on $\mathbb{R} \times \Theta$. The next approach, which we refer to as ‘‘Construction III’’, is based on Mellin transform of G_θ and applies to NEFs with separable moment sequences.

Theorem 5 *Assume that the NEF \mathcal{F} has a separable moment sequence $\{\tilde{c}_n(\theta)\}_{n \geq 0}$ at θ_0 , and let ω be admissible. For $t, x \in \mathbb{R}$ set*

$$K(t, x; \mu_0) = \frac{1}{\zeta(\theta_0)} \int_{[-1,1]} \sum_{n=0}^{\infty} \frac{(-tsx)^n \cos\left(\frac{\pi}{2}n + ts\xi(\theta_0)\right)}{\tilde{a}_n n!} \omega(s) ds. \quad (31)$$

Then

$$\psi(t, \mu; \mu_0) = \int K(t, x; \theta_0) dG_\theta(x) = \frac{\zeta(\theta)}{\zeta(\theta_0)} \int_{[-1,1]} \cos(ts(\xi(\theta_0) - \xi(\theta))) \omega(s) ds, \quad (32)$$

$\psi(t, \theta_0; \theta_0) = 1$ for any t and $\lim_{t \rightarrow \infty} \psi(t, \theta; \theta_0) = 0$ for each $\theta \neq \theta_0$.

Compared to Constructions I and II, Construction III involves the integral of an infinite series and is more complicated. However, it deals with NEFs that have more complicated structures than the former two.

4.5 Construction III: two examples

We provide two examples from Construction III for Exponential and Gamma families, respectively. Note that Gamma family contains Exponential family and Chi-square family.

Example 12 *Exponential family* Exponential(μ) with mean $\mu > 0$ and basis $\beta(dx) = e^{-x} \nu(dx)$, for which $L(\theta) = (1 - \theta)^{-1}$ and $\mu(\theta) = 1 - \theta$ for $\theta < 1$. Further,

$$\frac{dF_\mu}{d\nu}(x) = f_\mu(x) = \mu e^{-\mu x} 1_{[0, \infty)}(x)$$

and $\tilde{c}_n(\mu) = \mu \frac{n!}{\mu^n}$. So, $L(\mu) = \mu^{-1}$, $\xi(\mu) = \mu^{-1}$, $\tilde{a}_n = n!$ and $\zeta \equiv 1$. Setting

$$K(t, x; \mu_0) = \int_{[-1,1]} \omega(s) \sum_{n=0}^{\infty} \frac{(-tsx)^n \cos\left(\frac{\pi}{2}n + \frac{ts}{\mu_0}\right)}{(n!)^2} ds \quad (33)$$

gives

$$\psi(t, \mu; \mu_0) = \int K(t, x; \mu_0) dF_\mu(x) = \int_{[-1,1]} \cos(ts(\mu^{-1} - \mu_0^{-1})) \omega(s) ds.$$

Example 13 *Gamma family* Gamma(θ, σ) with basis β such that

$$\frac{d\beta}{d\nu}(x) = \frac{x^{\sigma-1} e^{-x}}{\Gamma(\sigma)} 1_{(0, \infty)}(x) dx \quad \text{with } \sigma > 0,$$

where Γ is the Euler's Gamma function. So, $L(\theta) = (1 - \theta)^{-\sigma}$,

$$\frac{dG_\theta}{d\nu}(x) = f_\theta(x) = (1 - \theta)^\sigma \frac{e^{\theta x} x^{\sigma-1} e^{-x}}{\Gamma(\sigma)} 1_{(0,\infty)}(x) \quad (34)$$

for $\theta < 1$, and $\mu(\theta) = \frac{\sigma}{1-\theta}$. Since

$$\tilde{c}_n(\theta) = (1 - \theta)^\sigma \int_0^\infty \frac{e^{-y} y^{n+\sigma-1}}{\Gamma(\sigma)} \frac{dy}{(1 - \theta)^{n+\sigma-1}} = \frac{\Gamma(n + \sigma)}{\Gamma(\sigma)} \frac{1}{(1 - \theta)^n},$$

we see $\xi(\theta) = (1 - \theta)^{-1}$, $\tilde{a}_n = \frac{\Gamma(n+\sigma)}{\Gamma(\sigma)}$ and $\zeta \equiv 1$. Setting

$$K(t, x; \theta_0) = \int_{[-1,1]} \omega(s) \sum_{n=0}^{\infty} \frac{(-tsx)^n \Gamma(\sigma) \cos\left(\frac{\pi}{2}n + \frac{ts}{1-\theta_0}\right)}{n! \Gamma(\sigma + n)} ds,$$

we obtain

$$\psi(t, \theta; \theta_0) = \int_{[-1,1]} \cos\left(ts \left((1 - \theta_0)^{-1} - (1 - \theta)^{-1}\right)\right) \omega(s) ds,$$

for which $\psi(t, \theta_0; \theta_0) = 1$ for all t and $\lim_{t \rightarrow \infty} \psi(t, \mu; \theta_0) = 0$ for each $\theta \neq \theta_0$.

Recall (12) of Construction I based on Fourier transform, i.e.,

$$\psi(t, \mu; \mu_0) = \int K(t, x; \mu_0) dF_\mu(x) = \int K(t, y + (\mu - \mu_0); \mu_0) dF_{\mu_0}(y),$$

where the action of Fourier transform is seen as the translation $K(t, x; \mu_0) \mapsto K(t, y + (\mu - \mu_0); \mu_0)$.

In contrast, the action of Mellin transform is seen via

$$\begin{aligned} \psi(t, \theta; \theta_0) &= \int K(t, x; \theta_0) dG_\theta(x) \\ &= \int_0^\infty K(t, x; \theta_0) (1 - \theta)^\sigma \frac{e^{-(1-\theta)x} x^{\sigma-1}}{\Gamma(\sigma)} dx \end{aligned} \quad (35)$$

$$= \int K\left(t, \frac{y}{1-\theta}; \theta_0\right) \beta(dy), \quad (36)$$

where from (35) to (36) scaling $K(t, x; \theta_0) \mapsto K\left(t, y(1 - \theta)^{-1}; \theta_0\right)$ is induced. This comparison clearly shows the action of Mellin transform as the multiplication-convolution equivalence, in contrast to the action of Fourier transform as the translation-convolution equivalence.

5 Two non-existence results for the Strategy

In this section, we provide two example families for which the Strategy is not implementable. To state them, we introduce

Definition 5 The proposition “PropK”: there exists a $K : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that K does not depend on any $\theta \in \Theta_1$ with $\theta \neq \theta_0$ and that

$$\psi(t, \theta; \theta_0) = \int K(t, x; \theta_0) dG_\theta(x)$$

satisfies $\lim_{t \rightarrow \infty} \psi(t, \theta_0; \theta_0) = 1$ and $\lim_{t \rightarrow \infty} \psi(t, \theta; \theta_0) = 0$ for $\theta \in \Theta_1$ with $\theta \neq \theta_0$, where Θ_1 is a subset of Θ that has a non-empty interior and does not contain θ_0 .

For the following two families, i.e., Inverse Gaussian and Binomial families, PropK does not hold. Note that the former family is discrete but has a finite support, whereas the latter is continuous but does not have a separable moment sequence.

Example 14 The Inverse Gaussian family $\text{InvGaussian}(\theta, \sigma)$ with scale parameter $\sigma > 0$ and basis

$$\beta(dx) = x^{-3/2} \exp\left(-\frac{\sigma^2}{2x}\right) \frac{\sigma}{\sqrt{2\pi}} 1_{(0, \infty)}(x) \nu(dx),$$

for which $L(\theta) = \exp(-\sigma\sqrt{-2\theta})$ with $\theta < 0$. So,

$$\frac{dG_\theta}{d\nu}(x) = f_\theta(x) = \frac{x^{-3/2}}{L(\theta)} \exp\left(-\frac{\sigma^2}{2x}\right) \frac{\sigma}{\sqrt{2\pi}} 1_{(0, \infty)}(x).$$

By a change of variables $y = \frac{\sigma^2}{2x}$, we obtain

$$\begin{aligned} \int K(t, x; \theta_0) dG_\theta(x) &= \frac{1}{L(\theta)} \frac{\sigma}{\sqrt{2\pi}} \int_0^\infty K(t, x; \theta_0) x^{-3/2} \exp\left(-\frac{\sigma^2}{2x}\right) dx \\ &= \frac{1}{\sqrt{\pi}L(\theta)} \int_0^\infty K\left(t, \frac{\sigma^2}{2y}; \theta_0\right) y^{-1/2} e^{-y} dy. \end{aligned} \quad (37)$$

Since the integral on the right hand side of (37) is not a function of θ for $\theta \neq \theta_0$, PropK does not hold and K does not exist. Note that the Gaussian family and Inverse Gaussian family are reciprocal pairs, called so by [Letac and Mora \(1990\)](#).

Example 15 Binomial family $\text{Binomial}(\theta, n)$ such that

$$\mathbb{P}(X = k) = \binom{n}{k} \theta^k (1 - \theta)^{n-k} \quad \text{for } \theta \in (0, 1).$$

We will show that if PropK holds, then K must be a function of $\theta \in \Theta_1$, a contradiction. Assume PropK holds. Then,

$$\psi(t, \theta; \theta_0) = \int K(t, x; \theta_0) dG_\theta(x) = (1 - \theta)^n \sum_{k=0}^n a_k(t; \theta_0) \binom{n}{k} \frac{\theta^k}{(1 - \theta)^k},$$

where $a_x(t; \theta_0) = K(t, x; \theta_0)$ for $x = 0, \dots, n$.

Let $d_k(t; \theta_0) = a_k(t; \theta_0) \binom{n}{k}$ and $q(\theta) = \frac{\theta}{1-\theta}$. Pick $n+1$ distinct values $\varpi_i, i = 0, \dots, n$ from $(0, 1)$ such that $\varpi_0 = \theta_0$ and $\varpi_i \in \Theta_1$ for $i = 1, \dots, n$. Further, define the $(n+1) \times (n+1)$ Vandermonde matrix \mathbf{V}_{n+1} whose (i, j) entry is $q^{j-1}(\varpi_{i-1})$, i.e., $\mathbf{V}_{n+1} = (q^{j-1}(\varpi_{i-1}))$, $\mathbf{d}_t = (d_0(t; \theta_0), \dots, d_n(t; \theta_0))^T$ and $\mathbf{b} = ((1-\theta_0)^{-n}, 0, \dots, 0)^T$. Then the determinant $|\mathbf{V}_{n+1}| \neq 0$, and the properties of K imply

$$\lim_{t \rightarrow \infty} \mathbf{V}_{n+1} \mathbf{d}_t = \mathbf{b}.$$

However, $\mathbf{V}_{n+1} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ as a bounded linear mapping is a homeomorphism with the bounded inverse \mathbf{V}_{n+1}^{-1} . So,

$$\lim_{t \rightarrow \infty} \mathbf{d}_t = \lim_{t \rightarrow \infty} \mathbf{V}_{n+1}^{-1} (\mathbf{V}_{n+1} (\mathbf{d}_t)) = \mathbf{V}_{n+1}^{-1} \mathbf{b}. \quad (38)$$

By Cramer's rule, we obtain

$$\lim_{t \rightarrow \infty} d_n(t; \theta_0) = \frac{(-1)^n |\mathbf{V}_n|}{|\mathbf{V}_{n+1}| (1-\theta_0)^n} = \frac{(-1)^n (1-\theta_0)^{-n}}{\prod_{k=1}^n (q(\varpi_k) - q(\theta_0))}, \quad (39)$$

where \mathbf{V}_n is the submatrix of \mathbf{V}_{n+1} obtained by removing the first row and last column of \mathbf{V}_{n+1} . But (39) is a contradiction since $d_n(t; \theta_0)$ does not depend on any $\theta \in \Theta_1$ for all t . For the case $n = 1$, we easily see from (39) the contradiction

$$\lim_{t \rightarrow \infty} a_1(t; \theta_0) = \frac{-(1-\theta_0)^{-1}}{q(\theta) - q(\theta_0)} \quad \text{for each } \theta \in \Theta_1.$$

To summarize, PropK does not hold and K does not exist.

6 Construction II: consistency and speed of convergence

Recall $H^{(k)}(0) = c_k k!$ for $k \in \mathbb{N}$. Call the sequence $\{(c_k k!)^{-1}, k \in \mathbb{N}\}$ the ‘‘reciprocal derivative sequence (of H at 0)’’. The following lemma gives bounds on the magnitudes of this sequence for the examples given in Section 4.3. It will help derive concentration inequalities for estimators from Construction II.

Lemma 2 Consider the examples given in Section 4.3. Then $c_k k! \equiv 1$ for Poisson family, whereas for Negative Binomial, Abel and Takács families with a fixed n and $\sigma > 0$,

$$\frac{1}{c_k k!} \leq \frac{C}{k!} \quad \text{for all } k \in \mathbb{N}. \quad (40)$$

However, for Strict Arcsine and Large Arcsine families, both with a fixed $\sigma > 0$, (40) does not hold. On the other hand, for any $\tilde{r} > 0$ smaller than the radius R_H of convergence of H ,

$$\frac{1}{c_k k!} = \frac{1}{H^{(k)}(0)} \geq \frac{1}{H(\tilde{r})} \frac{\tilde{r}^k}{k!} \quad \text{for all } k \in \mathbb{N}. \quad (41)$$

Lemma 2 shows that, among the six discrete NEFs with support \mathbb{N} given in [Section 4.3](#), the reciprocal derivative sequence for Poisson family has the largest magnitude, whereas this sequence for Negative Binomial, Abel and Takács families with a fixed n and $\sigma > 0$ are all dominated by the “reciprocal factorial sequence” $\{\frac{1}{k!} : k \in \mathbb{N}\}$ approximately. Further, [Lemma 2](#) asserts that the reciprocal derivative sequence dominates the “exponential sequence” $\{H(\tilde{r}) \frac{k!}{\tilde{r}^k} : k \in \mathbb{N}\}$ for any $\tilde{r} > 0$ smaller than the radius of convergence of H .

Let $\eta = e^\theta$ for $\theta \in \Theta$, $\eta_i = e^{\theta_i}$ for $0 \leq i \leq m$ and $\boldsymbol{\eta} = (\eta_1, \dots, \eta_m)$. First, we provide upper bounds on the variance and oscillations of $\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})$ when t is positive and sufficiently large.

Theorem 6 *Let \mathcal{F} be an NEF generated by β in (27), $\{z_i\}_{i=1}^m$ independent with CDFs $\{G_{\theta_i}\}_{i=1}^m$ belonging to \mathcal{F} , λ a positive constant, and t positive and sufficiently large.*

1. Let $\phi_m(L, \boldsymbol{\eta}) = \min_{1 \leq i \leq m} \{L(\theta_i) \eta_i^{1/4}\}$. If (40) holds, then

$$\mathbb{V}[\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})] \leq V_{\text{II},1}^{(m)} = \frac{C \exp(2t \|\boldsymbol{\eta}\|_\infty^{1/2})}{m \sqrt{t} \phi_m(L, \boldsymbol{\theta})}$$

and

$$\Pr(|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})| \geq \lambda) \leq \lambda^{-2} V_{\text{II},1}^{(m)}. \quad (42)$$

2. Let $L_{\min}^{(m)} = \min_{1 \leq i \leq m} L(\theta_i)$. Then for Poisson family,

$$\mathbb{V}[\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})] \leq V_{\text{II},2}^{(m)} = \frac{C \exp(t^2 \|\boldsymbol{\eta}\|_\infty^{1/2})}{m L_{\min}^{(m)}} \quad (43)$$

and

$$\Pr(|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})| \geq \lambda) \leq \lambda^{-2} V_{\text{II},2}^{(m)}.$$

We remark that the assertion in [Theorem 6](#) on Poisson family holds for any NEF with support \mathbb{N} such that $c_k k! \leq C$ for all $k \in \mathbb{N}$. With [Theorem 6](#), we derive the uniform consistency classes and speeds of convergence for the estimators from Construction II. Recall $\eta = e^\theta$ for $\theta \in \Theta$ and $\boldsymbol{\eta} = (e^{\theta_1}, \dots, e^{\theta_m})$.

Theorem 7 *Let \mathcal{F} be the NEF generated by β in (27), $\{z_i\}_{i=1}^m$ independent with CDFs $\{G_{\theta_i}\}_{i=1}^m$ belonging to \mathcal{F} , and ρ a finite, positive constant.*

1. If (40) holds, then a uniform consistency class is

$$\mathcal{Q}_{\text{II},1}(\boldsymbol{\theta}, t, \pi_{1,m}; \gamma) = \left\{ \begin{array}{l} \|\boldsymbol{\theta}\|_\infty \leq \rho, \pi_{1,m} \geq m^{(\gamma-1)/2}, t = 2^{-1} \|\boldsymbol{\eta}\|_\infty^{-1/2} \gamma \log m, \\ \lim_{m \rightarrow \infty} t \min_{i \in I_{1,m}} |\eta_0 - \eta_i| = \infty \end{array} \right\}$$

for any fixed $\gamma \in (0, 1]$. The speed of convergence is poly-log.

2. For Poisson family, a uniform consistency class is

$$\mathcal{Q}_{\text{II},2}(\boldsymbol{\theta}, t, \pi_{1,m}; \gamma) = \left\{ \begin{array}{l} \|\boldsymbol{\theta}\|_{\infty} \leq \rho, \pi_{1,m} \geq m^{(\gamma'-1)/2}, t = \sqrt{\|\boldsymbol{\eta}\|_{\infty}^{-1/2} \gamma \log m}, \\ \lim_{m \rightarrow \infty} t \min_{i \in I_{1,m}} |\eta_0 - \eta_i| = \infty \end{array} \right\}$$

for any fixed $\gamma \in (0, 1)$ and $\gamma' > \gamma$. The speed of convergence is poly-log.

In [Theorem 7](#), the speed of convergence and uniform consistency class depend on $\|\boldsymbol{\eta}\|_{\infty}$ and $\min_{i \in I_{1,m}} |\eta_0 - \eta_i|$, whereas they only depend on $\min_{i \in I_{1,m}} |\mu_i - \mu_0|$ for location-shift families. However, when the GF H of the basis β has finite radius of convergence, their dependence on $\|\boldsymbol{\eta}\|_{\infty}$ can be removed, as justified by:

Corollary 5 *Let \mathcal{F} be the NEF generated by β in (27), $\{z_i\}_{i=1}^m$ independent with CDFs $\{G_{\theta_i}\}_{i=1}^m$ belonging to \mathcal{F} , and ρ a finite, positive constant. If (40) holds and H has a finite radius of convergence R_H , then for positive and sufficiently large t*

$$\mathbb{V}[\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})] \leq \frac{C \exp(2tR_H^{1/2})}{m \sqrt{t} \phi_m(L, \boldsymbol{\theta})},$$

and a uniform consistency class is

$$\tilde{\mathcal{Q}}_{\text{II},1}(\boldsymbol{\theta}, t, \pi_{1,m}; \gamma) = \left\{ \begin{array}{l} \|\boldsymbol{\theta}\|_{\infty} \leq \rho, t = 2^{-1} R_H^{-1/2} \gamma \log m, \\ \pi_{1,m} \geq m^{(\gamma-1)/2}, \min_{i \in I_{1,m}} |\eta_0 - \eta_i| \geq \frac{\log \log m}{\sqrt{2} \log m} \end{array} \right\}$$

for any $\gamma \in (0, 1]$.

Since $\sup \Theta \leq \log R_H$ and $\|\boldsymbol{\eta}\|_{\infty} \leq R_H$, the proof [Corollary 5](#) follows easily from the first claims of [Theorem 6](#) and [Theorem 7](#) and is omitted. The uniform consistency class $\tilde{\mathcal{Q}}_{\text{II},3}(\boldsymbol{\theta}, t, \pi_{1,m}; \gamma)$ is fully data-adaptive and only requires information on $\min_{i \in I_{1,m}} |\eta_0 - \eta_i|$, as do Jin's estimator of [Jin \(2008\)](#) for Gaussian family and Construction I for location-shift families on $\min_{i \in I_{1,m}} |\mu_i - \mu_0|$.

7 Construction III: consistency and speed of convergence

We will focus on Gamma family and show that the corresponding estimators are uniformly consistent. Recall f_{θ} defined by (34) for Gamma family. Then $f_{\theta}(x) = O(x^{\sigma-1})$ as $x \rightarrow 0+$, which tends to 0 when $\sigma > 1$. The next result provides upper bounds on the variance and oscillations of $\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})$ when t is positive and large.

Theorem 8 Consider Gamma family such that $\{z_j\}_{j=1}^m$ are independent with parameters $\{(\theta_i, \sigma)\}_{i=1}^m$ and a fixed $\sigma > 0$. Assume t is positive and sufficiently large and set $u_{3,m} = \min_{1 \leq i \leq m} \{1 - \theta_i\}$.

Then

$$\mathbb{V}[\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})] \leq V_{\text{III}}^{(m)} = \frac{C}{m} \exp\left(\frac{4t}{u_{3,m}}\right) \sum_{i=1}^m \left(\frac{t}{1 - \theta_i}\right)^{3/4 - \sigma}$$

and

$$\Pr(|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})| \geq \lambda) \leq \lambda^{-2} V_{\text{III}}^{(m)}. \quad (44)$$

Recall $u_{3,m} = \min_{1 \leq i \leq m} \{1 - \theta_i\}$ and $\xi(\theta) = (1 - \theta)^{-1}$, such that $u_{3,m} = \min_{1 \leq i \leq m} \xi^{-1}(\theta_i)$. Using [Theorem 8](#), we show the uniform consistency and speed of convergence of the estimator for Gamma family.

Theorem 9 Consider Gamma family such that $\{z_j\}_{j=1}^m$ are independent with parameters $\{(\theta_i, \sigma)\}_{i=1}^m$ and a fixed $\sigma > 0$. Let $\rho > 0$ be a finite constant. If $\sigma > 3/4$, then the uniform consistency class

$$\mathcal{Q}_{\text{III}}(\boldsymbol{\theta}, t, \pi_{1,m}; \gamma) = \left\{ \begin{array}{l} \|\boldsymbol{\theta}\| \leq \rho, t = 4^{-1} \gamma u_{3,m} \log m, \lim_{m \rightarrow \infty} u_{3,m} \log m = \infty, \\ \pi_{1,m} \geq m^{(\gamma-1)/2}, \lim_{m \rightarrow \infty} t \min_{i \in I_{1,m}} |\xi(\theta_0) - \xi(\theta_i)| = \infty \end{array} \right\}$$

for any fixed $\gamma \in (0, 1]$. On the other hand, if $\sigma \leq 3/4$, then

$$\mathcal{Q}_{\text{III}}(\boldsymbol{\theta}, t, \pi_{1,m}; \gamma) = \left\{ \begin{array}{l} \|\boldsymbol{\theta}\| \leq \rho, t = 4^{-1} \gamma u_{3,m} \log m, \pi_{1,m} \geq m^{(\gamma'-1)/2}, \\ \lim_{m \rightarrow \infty} t \min_{i \in I_{1,m}} |\xi(\theta_0) - \xi(\theta_i)| = \infty \end{array} \right\}$$

for any fixed $\gamma \in (0, 1)$ and $\gamma' > \gamma$. However, in either case, the speed of convergence is poly-log.

In [Theorem 9](#), the speed of convergence and uniform consistency class depend on $u_{3,m} = \min_{1 \leq i \leq m} \{1 - \theta_i\}$ and $\xi_{3,m} = \min_{i \in I_{1,m}} |\xi(\theta_0) - \xi(\theta_i)|$. Since $\theta < 1$ for Gamma family, $u_{3,m}$ measures how close a G_{θ_i} is to the singularity where a Gamma density is undefined, and it is sensible to often assume $\liminf_{m \rightarrow \infty} u_{3,m} > 0$. On the other hand, $\sigma \xi(\theta) = \mu(\theta)$ for all $\theta \in \Theta$ for Gamma family. So, $\xi_{3,m}$ measures the minimal difference between the means of G_{θ_i} for $i \in I_{1,m}$ and G_{θ_0} , and $\xi_{3,m}$ cannot be too small relative to t as $t \rightarrow \infty$ in order for the estimator to achieve consistency.

8 Simulation studies

We present a simulation study on $\hat{\varphi}_m(t_m, \mathbf{z})$, with comparison to the ‘‘MR’’ estimator of [Meinshausen and Rice \(2006\)](#). Since the MR estimator is only applicable to p-values that have continuous distributions, its performance, when applied to discrete p-values such as those induced by Poisson and Negative Binomial distributions (to be considered hereunder), provides information on its robustness. Specifically, when X_0 is a realization of a Poisson or Negative Binomial random variable with CDF F_0^* , its p-value is $F_0^*(X_0)$.

8.1 Simulation design

For $a < b$, let $\text{Unif}(a, b)$ be the uniform random variable or the uniform distribution on the closed interval $[a, b]$. We consider 7 values for m as 10^3 , 5×10^3 , 10^4 , 5×10^4 , 10^5 , 5×10^5 or 10^6 , and 4 sparsity levels for $\pi_{1,m}$, i.e., the dense regime $\pi_{1,m} = 0.2$, moderately sparse regime $\pi_{1,m} = m^{-0.2}$, critically sparse regime $\pi_{1,m} = m^{-0.5}$ and very sparse regime $\pi_{1,m} = m^{-0.7}$, where we recall $\pi_{1,m} = 1 - m_0 m^{-1}$ and $m_0 = |\{1 \leq i \leq m : \mu_i = \mu_0\}|$. Further, we consider 5 distribution families \mathcal{F} , i.e., Laplace, Cauchy, Poisson, Negative Binomial and Chi-square families, set as follows:

- For Laplace and Cauchy families, $\sigma = 1$ and $\mu_0 = 0$ is set, and the nonzero μ_i 's are generated independently such that their absolute values $|\mu_i|$ are from $\text{Unif}(0.75, 5)$ but each μ_i has probability 0.5 to be negative or positive.
- For Negative Binomial family, $n = 5$ and $\theta_0 = -4.5$, for Poisson family $\theta_0 = 0.08$, and for Gamma family $\theta_0 = 0.5$ and $\sigma = 6$. Note that for Gamma family, under the null hypothesis the corresponding distribution is a Chi-square distribution with $2^{-1}\sigma$ degrees of freedom. For all three families, $\theta_i = \theta_0 \rho_i$ for $i = m_0 + 1, \dots, m_0 + \lfloor 2^{-1} m \pi_m \rfloor$ and $\theta_i = \theta_0 \rho_i^{-1}$ for $i = m_0 + \lfloor 2^{-1} m \pi_m \rfloor + 1, \dots, m$, where $\{\rho_i\}_{i=m_0+1}^m$ are independently generated from $\text{Unif}(10, 13)$ for Poisson family, from $\text{Unif}(8, 15)$ for Negative Binomial family, and from $\text{Unif}(1.2, 1.5)$ for Gamma family.

Each triple (m, π_m, \mathcal{F}) gives an experiment, and there are a total of 140 experiments. For the estimator $\hat{\varphi}_m(t_m, \mathbf{z})$, the sequence $\{t_m\}_{m \geq 1}$ is set to be that given by [Corollary 3](#) for Gaussian and Laplace families, by [Theorem 7](#) for Poisson family, by [Corollary 5](#) for Negative Binomial family, and by [Theorem 9](#) for Gamma family, where the tuning parameter γ is set to be the supremum of its feasible range to enable the estimator to achieve a fast speed of convergence. For Gamma family, the power series used to construct the estimator is replaced by the partial sum of its first 21 terms. Each experiment is repeated independently 250 times so that summary statistics can be obtained. For an estimator $\hat{\pi}_{1,m}$ of $\pi_{1,m}$ for $m \geq 1$ for each experiment, the mean and standard deviation of $\tilde{\delta}_m = \hat{\pi}_{1,m} \pi_{1,m}^{-1} - 1$ is estimated from the 250 realizations.

8.2 Simulation results

For an estimator $\hat{\pi}_{1,m}$ of $\pi_{1,m}$, we will measure its stability by the standard deviation σ_m^* of $\tilde{\delta}_m$ and its accuracy by the mean μ_m^* of $\tilde{\delta}_m$. Among two estimators for a fixed m , the one that has both smaller σ_m^* and $|\mu_m^*|$ will be considered better. The supplementary material contains boxplots that summarize the performances of the two estimators.

The following five observations have been made: (1) For Laplace and Cauchy families, the new estimator is very accurate and much better than the MR estimator. In the dense and moderately sparse regimes, there is very strong evidence on the consistency of the new estimator

since $\tilde{\delta}_m$ displays a strong trend to converge to 0. These may hold true when the new estimator is applied to other location-shift families. (2) For Poisson and Negative Binomial families, the new estimator is accurate and much better than the MR estimator. In the dense and moderately sparse regimes, there is strong evidence on the convergence of $\tilde{\delta}_m$ (even though not necessarily to 0) when the new estimator is applied to Negative Binomial family, whereas there is no strong evidence of convergence of $\tilde{\delta}_m$ when it is applied to Poisson family. In other words, we have not observed strong evidence on the consistency of the new estimator. This may be a consequence of non-adaptively choosing t_m for the estimator, and is worth further investigation. In contrast, the MR estimator is almost always zero, i.e., it is rarely able to detect the existence of false null hypotheses. (3) For Gamma family, the new estimator is more accurate than the MR estimator, and it often severely underestimates $\pi_{1,m}$. When the new estimator is applied to the dense and moderately sparse regimes, there is strong evidence on the convergence of $\tilde{\delta}_m$ but there is no strong evidence on the consistency of the new estimator. This may be due to truncating the power series in Construction III when implementing the estimator for Gamma family and non-adaptively choosing t_m for the estimator, and requires further investigation. In contrast, the MR estimator is almost always zero, often failing to detect the existence of false null hypotheses. Such an interesting behavior for the MR estimator has not been reported before. (4) The MR estimator, if not being 0 for all almost all repetitions of an experiment, is less stable than the new estimator and can be much so when m is large. In the critically sparse and very sparse regimes, the new estimator does not seem to be consistent and its $\tilde{\delta}_m$ does not display a trend of convergence as m increases. Similarly, in these regimes, the MR estimator does not seem to be consistent unless it is identically zero. However, this does not contradict the theory for the MR estimator since its consistency requires that p-values under the alternative hypothesis be identically distributed and they are not so in the simulation study here. (5) In the dense and moderately sparse regimes, the new estimator usually underestimates $\pi_{1,m}$, i.e., the estimated proportion of true null hypotheses, $\hat{\pi}_{0,m}$, induced by the estimator is usually conservative. This is appealing in that a one-step adaptive FDR procedure that employs $\hat{\pi}_{0,m}$ is usually conservative.

9 Discussion

We have demonstrated that solutions of Lebesgue-Stieltjes integral equations can serve as a universal construction for proportions estimators, provided proportion estimators for random variables with three types of distributions, and justified under independence the uniform consistency and speeds of convergence of the estimators. In order to achieve uniform consistency for estimators from Construction II and III, we may need to estimate the supremum norm of the parameter vector or the infimum of a transform of the vector itself. In an accompanying article, we will deal with this issue, discuss how truncating the power series in Construction III affects the accuracy of the induced estimators, how to adaptively determine the tuning parameter for

the proposed estimators, and report via extensive simulation studies on their non-asymptotic performances.

Our work induces three topics that are worthy of future investigations. Firstly, we have only considered estimating the proportion of parameters that are unequal to a fixed value, i.e., the proportion induced by the functional that maps a parameter to a fixed value. It would be interesting to construct uniformly consistent estimators of proportions induced by other functionals. Further, we have only considered independent random variables. Extending the consistency results provided here to dependent case will greatly enlarge the scope of applications of the estimators, as did by [Chen \(2018\)](#) and [Jin and Cai \(2007\)](#) to Jin’s estimator of [Jin \(2008\)](#) for Gaussian family and Gaussian mixtures. Moreover, Construction I, II and III are applicable to random variables whose distributions are from different sub-families of the same type of distributions, and results on the uniform consistency of the corresponding proportion estimators can be extended to this case. Finally, following the principles in Section 3 of [Jin \(2008\)](#), Construction I, II and III can possibly be applied to consistently estimate the mixing proportions for two-component mixture models at least one of whose components follows a distribution discussed in this work.

Secondly, we have only been able to construct proportion estimators for three types of distributions, and provide Gamma family as an example for Construction III. It is worthwhile to explore other settings for which solutions of Lebesgue-Stieltjes integro-differential equations exist, can be analytically expressed, and serve as consistent proportion estimators. Further, we have not studied optimal properties of the proposed estimators, and with regard to this the techniques of [Cai and Jin \(2010\)](#) and [Carpentier and Verzelen \(2017\)](#) may be useful.

Thirdly, we have introduced the concept of “the family of distributions with Riemann-Lebesgue type characteristic functions (RL type CFs)” (see [Definition 1](#)) for which

$$\left\{ t \in \mathbb{R} : \hat{F}_{\mu_0}(t) = 0 \right\} = \emptyset, \quad (45)$$

and shown that it contains several location-shift families. The requirement (45) precludes the characteristic function \hat{F}_{μ_0} to have any real zeros. We are aware that Poisson family is infinitely divisible but not a location-shift family and does not have RL type CFs. However, it is unclear to us the relationships (with respect to set inclusion) between infinitely divisible distributions, location-shift families and distributions with RL type CFs. So, a better understanding of such relationships will contribute both to the theory of probability distributions and finding examples different than those given here that Constructions I, II and III apply to.

Acknowledgements

Part of the research was funded by the New Faculty Seed Grant provided by Washington State University. I am very grateful to Gérard Letac for his guidance on my research involving natural

exponential families and constant encouragements, Jiashun Jin for providing the technical report [Jin \(2006\)](#), warm encouragements and comments on the presentation of an earlier version of the manuscript, Mark D. Ward for comments on exponential generating functions, Kevin Vixie, Hong-Ming Yin and Charles N. Moore for discussions on solution of integro-differential equations, Sheng-Chi Liu for discussions on solutions of algebraic equations, and Ovidiu Costin and Sergey Lapin for help with access to two papers.

References

- Auer, P. and Doerge, R. (2010). Statistical design and analysis of RNA-Seq data, *Genetics* (185): 405–416.
- Bar-Lev, S., Boukai, B. and Landsman, Z. (2016). The Kendal-Ressel exponential dispersion model: Some statistical aspects and estimation, *International Journal of Statistics and Probability* **5**(3).
- Benjamini, Y. and Yekutieli, D. (2001). The control of the false discovery rate in mutliple testing under dependency, *Ann. Statist.* **29**(4): 1165–1188.
- Blanchard, G. and Roquain, E. (2009). Adaptive false discovery rate control under independence and dependence, *J. Mach. Learn. Res.* **10**: 2837–2871.
- Cai, T. T. and Jin, J. (2010). Optimal rates of convergence for estimating the null density and proportion of nonnull effects in large-scale multiple testing, *Ann. Statist.* **38**(1): 100–145.
- Cai, T. T. and Sun, W. (2009). Simultaneous testing of grouped hypotheses: Finding needles in multiple haystacks, *J. Amer. Statist. Assoc.* **104**(488): 1467–1481.
- Carpentier, A. and Verzelen, N. (2017). Adaptive estimation of the sparsity in the gaussian vector model, [arXiv:1703.00167](#) .
- Chen, X. (2018). Consistent FDR estimation for adaptive multiple testing Normal means under principal correlation structure, [arXiv:1410.4275v4](#) .
- Chen, X. and Doerge, R. W. (2017). A weighted FDR procedure under discrete and heterogeneous null distributions, [arXiv:1502.00973v4](#) .
- Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J. and Knuth, D. E. (1996). On the lambertw function, *Adv. Comput. Math.* **5**(1): 329–359.
- Costin, O., Falkner, N. and McNeal, J. D. (2016). Some generalizations of the RiemannLebesgue lemma, *Am. Math. Mon.* **123**(4): 387–391.

- Efron, B., Tibshirani, R., Storey, J. D. and Tusher, V. (2001). Empirical bayes analysis of a microarray experiment, *J. Amer. Statist. Assoc.* **96**(456): 1151–1160.
- Fischer, M. J. (2014). *Generalized Hyperbolic Secant Distributions*, Springer.
- Genovese, C. and Wasserman, L. (2004). A stochastic process approach to false discovery control, *Ann. Statist.* **32**(3): 1035–1061.
- Gilbert, P. B. (2005). A modified false discovery rate multiple-comparisons procedure for discrete data, applied to human immunodeficiency virus genetics, *J. R. Statist. Soc. Ser. C* **54**(1): 143–158.
- Gronwall, T. H. (1932). An inequality for the bessel functions of the first kind with imaginary argument, *Ann. Math.* **33**(2): 275–278.
- Hoeffding, W. (1963). Probability inequalities for sums of bounded random variables, *J. Amer. Statist. Assoc.* **58**(301): 13–30.
- Jin, J. (2006). Proportions of nonzero normal means: universal oracle equivalence and uniformly consistent estimations, *Technical report. Department of statistics, Purdue University, West Lafayette* .
- Jin, J. (2008). Proportion of non-zero normal means: universal oracle equivalences and uniformly consistent estimators, *J. R. Statist. Soc. Ser. B* **70**(3): 461–493.
- Jin, J. and Cai, T. T. (2007). Estimating the null and the proportion of nonnull effects in large-scale multiple comparisons, *J. Amer. Statist. Assoc.* **102**(478): 495–506.
- Jin, J., Peng, J. and Wang, P. (2010). A generalized fourier approach to estimating the null parameters and proportion of nonnull effects in large-scale multiple testing, *J. Stat. Res.* **44**(1): 103–107.
- Kim, J. H. and Vu, V. H. (2000). Concentration of multivariate polynomials and its applications, *Combinatorica* **20**(3): 417–434.
- Kumar, P. R. and Bodhisattva, S. (2016). Estimation of a two-component mixture model with applications to multiple testing, *J. R. Statist. Soc. Ser. B* **78**(4): 869–893.
- Letac, G. (1992). *Lectures on natural exponential families and their variance functions*, Monografias de matemática, 50, IMPA, Rio de Janeiro.
- Letac, G. and Mora, M. (1990). Natural real exponential families with cubic variance functions, *Ann. Statist.* **18**(1): 1–37.
- Liu, Y., Sarkar, S. K. and Zhao, Z. (2016). A new approach to multiple testing of grouped hypotheses, *J. Stat. Plan. Inference* **179**: 1–14.

- Lukacs, E. (1970). *Characteristic Functions*, Hafner Publishing Company.
- Luo, S. and Zhang, Z. (2004). Estimating the first zero of a characteristic function, *Comptes Rendus Mathematique* **338**(3): 203–206.
- Mehrotra, D. V. and Heyse, J. F. (2004). Use of the false discovery rate for evaluating clinical safety data, *Stat. Methods Med. Res.* **13**(3): 227–238.
- Meinshausen, N. and Rice, J. (2006). Estimating the proportion of false null hypotheses among a large number of independently tested hypotheses, *Ann. Statist.* **34**(1): 373–393.
- Morris, C. N. (1982). Natural exponential families with quadratic variance functions, *Ann. Statist.* **10**(1): 65–80.
- Olver, F. W. J. (1974). *Asymptotics and special functions*, Academic Press, Inc., New York.
- Pitman, J. and Yor, M. (2003). Infinitely divisible laws associated with hyperbolic functions, *Canad. J. Math.* **55**(2): 292–330.
- Ploner, A., Calza, S., Gusnanto, A. and Pawitan, Y. (2006). Multidimensional local false discovery rate for microarray studies, *Bioinformatics* **22**(5): 556–565.
- Robinson, M. D. and Smyth, G. K. (2008). Small-sample estimation of negative binomial dispersion, with applications to SAGE data, *Biostatistics* **9**(2): 321–332.
- Sarkar, S. K. (2006). False discovery and false nondiscovery rates in single-step multiple testing procedures, *Ann. Statist.* **34**(1): 394–415.
- Schudy, W. and Sviridenko, M. (2012). Concentration and moment inequalities for polynomials of independent random variables, *Proceedings of the Twenty-third Annual ACM-SIAM Symposium on Discrete Algorithms, SODA '12*, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, pp. 437–446.
- Storey, J. (2003). The positive false discovery rate: a Bayesian interpretation and the q-value, *Ann. Statist.* **3**(6): 2013–2035.
- Storey, J. D. (2002). A direct approach to false discovery rates, *J. R. Statist. Soc. Ser. B* **64**(3): 479–498.
- Storey, J. D. (2007). The optimal discovery procedure: a new approach to simultaneous significance testing, *J. R. Statist. Soc. Ser. B* **69**(3): 347–368.
- Storey, J. D., Taylor, J. E. and Siegmund, D. (2004). Strong control, conservative point estimation in simultaneous conservative consistency of false discover rates: a unified approach, *J. R. Statist. Soc. Ser. B* **66**(1): 187–205.

- Swanepoel, J. W. H. (1999). The limiting behavior of a modified maximal symmetric $2s$ -spacing with applications, *Ann. Statist.* **27**(1): 24–35.
- Szegő, G. (1975). *Orthogonal polynomials*, American Mathematical Society, New York.
- Talagrand, M. (1996). Majorizing measures: the generic chaining, *Ann. Probab.* **24**(3): 1049–1103.
- Vu, V. H. (2002). Concentration of non-lipschitz functions and applications, *Random Structures & Algorithms* **20**(3): 262–316.
- Whittaker, E. and Watson, G. (1940). *A Course of Modern Analysis*, Cambridge University Press.

Supplementary material for “Estimators of the proportion of false null hypotheses: I”

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Abstract

We will discuss in [Appendix A](#) Construction III for Ressel and Hyperbolic Cosine families and in [Appendix B](#) uniform consistency in frequency domain for Constructions II and III and its relation to concentration inequalities for non-Lipschitz functions of independent random variables. Proofs related to Construction I, II and III are provided respectively in [Appendix C](#), [Appendix D](#) and [Appendix E](#). Additional simulation results are given in [Appendix F](#).

A On Construction III for Ressel and Hyperbolic Cosine families

For Ressel and Hyperbolic Cosine families, which are non-location-shift NEF-CVFs, we suspect that Construction III cannot be implemented based on the following initial results on their moment sequences.

Example 16 *Ressel family with basis*

$$\frac{d\beta}{d\nu}(x) = f(x) = \frac{\sigma x^{x+\sigma-1} e^{-x}}{\Gamma(x+\sigma+1)} 1_{(0,\infty)}(x) \quad \text{for } \sigma > 0 \quad (46)$$

and variance function $V(\mu) = \frac{\mu^2}{\sigma} (1 + \frac{\mu}{\sigma})$ for $\mu > 0$. Note that $\int_0^\infty \beta(dx) = 1$ by Proposition 5.5 of [Letac and Mora \(1990\)](#). From [Bar-Lev et al. \(2016\)](#) and references therein, we know the following: the Laplace transform $L_\sigma(\theta)$ of β cannot be explicitly expressed in θ ; $L_1(\theta) = \exp(-\tilde{\eta}(-\theta))$ where $\tilde{\eta}(-\theta)$ is the solution to the functional equation

$$\tilde{\eta}(-\theta) = \log(1 + \tilde{\eta}(-\theta) - \theta) \quad \text{with } \theta \leq 0;$$

$L_\sigma(\theta) = (L_1(\theta))^\sigma$, $\theta(\mu) = \log \frac{1+\mu}{\mu} - \mu^{-1}$ and $L_1(\theta(\mu)) = \frac{\mu}{1+\mu}$; $\lim_{x \rightarrow 0^+} \frac{d\beta}{d\nu}(x) = 0$ when $\sigma > 1$.

Let us compute $\tilde{c}_n^* = \int x^n e^{x\theta} \beta(dx)$ for $n \in \mathbb{N}$. Recall Hankel's formula for the reciprocal Gamma function, i.e.,

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_{\mathcal{C}} (-t)^{-z} e^{-t} dt \quad \text{with } \Re(z) > 0,$$

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where \mathcal{C} is the *Hankel contour* that wraps the non-negative real axis counterclockwise once and the logarithm function \log is such that $\log(-t) \in \mathbb{R}$ for $t < 0$; see, e.g., Section 12.22 of *Whittaker and Watson (1940)* for details on this. Then, for $n \geq 1$ we obtain

$$\frac{x^{x+\sigma}}{\Gamma(x+\sigma+1)} = \frac{\iota}{2\pi} \int_{\mathcal{C}} (-t)^{-x-\sigma-1} e^{-xt} dt \quad \text{for } x > 0$$

and

$$\begin{aligned} \tilde{c}_n^* &= \int_0^\infty \frac{\sigma x^{x+\sigma+n-1}}{\Gamma(x+\sigma+1)} e^{-x(1-\theta)} dx \\ &= \int_0^\infty \sigma x^{n-1} \exp(-x(1-\theta+t+\log(-t))) dx \frac{\iota}{2\pi} \int_{\mathcal{C}} (-t)^{-\sigma-1} dt \\ &= \frac{\iota \sigma (n-1)!}{2\pi} \int_{\mathcal{C}} \frac{(-t)^{-\sigma-1}}{(1-\theta+t+\log(-t))^n} dt. \end{aligned} \quad (47)$$

Let $b(\theta)$ be the lower real branch of the solutions in t to the functional equation

$$1 - \theta + t + \log(-t) = 0.$$

Then $b(\theta)$ is the Lambert W function $W_{-1}(\tilde{z})$ with $\tilde{z} = -\exp(\theta-1)$ and domain $\tilde{z} \in [-e^{-1}, 0)$ such that $W_{-1}(\tilde{z})$ decreases from $W_{-1}(-e^{-1}) = -1$ to $W_{-1}(0-) = -\infty$; see, e.g., *Corless et al. (1996)* for details on this. When $\theta < 0$, $b(\theta) > -1$. Since $b(\theta)$ is a pole of order n for the integrand $\tilde{R}(t)$ in (47), the residue theorem implies

$$\tilde{c}_n^* = -\sigma \lim_{t \rightarrow b(\theta)} \frac{d^{n-1}}{dt^{n-1}} \left((t - b(\theta))^n \tilde{R}(t) \right).$$

In particular,

$$\tilde{c}_1^* = -\sigma \frac{(-b(\theta))^{-\sigma-1}}{1+b^{-1}(\theta)} = -\sigma \frac{(-1)^{-\sigma-1} W_{-1}^{-\sigma}(-\exp(\theta-1))}{1+W_{-1}(-\exp(\theta-1))},$$

and \tilde{c}_n^* is a complicated function in $W_{-1}(-\exp(\theta-1))$ when n is large. So, Ressel family is unlikely to have a separable moment sequence.

Example 17 *Hyperbolic Cosine family with basis*

$$\frac{d\beta}{d\nu}(x) = \frac{2^{\sigma-2}}{\pi\Gamma(\sigma)} \left| \Gamma\left(\frac{\sigma}{2} + \iota\frac{x}{2}\right) \right|^2 \quad \text{for } \sigma > 0 \text{ and } x \in \mathbb{R}$$

and Fourier transform $L(\iota t) = (\cosh t)^{-\sigma}$ as shown on page 28 of *Letac and Mora (1990)*. So,

$$L(\theta) = 2^\sigma \left(e^{-\iota\theta} + e^{-\iota\theta} \right)^{-\sigma} = (\cos \theta)^{-\sigma} \quad \text{for } |\theta| < 2^{-1}\pi,$$

$\mu(\theta) = \sigma \tan \theta$ and $V(\mu) = \sigma \left(1 + \frac{\mu^2}{\sigma^2} \right)$ for $\mu \in \mathbb{R}$. From Theorems 2 and 3 of *Morris (1982)*,

we see that $\tilde{c}_n^* = \int x^n e^{\theta x} \beta(dx)$ for $n \geq 3$ is a polynomial of degree n in μ with at least one non-zero term of order between 1 and $n - 1$. So, Hyperbolic Cosine family is unlikely to have a separable moment sequence.

B On uniform consistency in frequency domain for Construction II and III

For Construction I applied to location-shift families with RL type CFs, we have proved

$$\Pr \left(\sup_{\boldsymbol{\mu} \in \mathcal{B}_m(\rho)} \left| \pi_{1,m}^{-1} \sup_{t \in [0, \tau_m]} \hat{\varphi}_m(t, \mathbf{z}) - 1 \right| \rightarrow 0 \right) \rightarrow 1,$$

for which the estimator is also consistent uniformly in $t \in [0, \tau_m]$ for a positive, increasing sequence $\tau_m \rightarrow \infty$. This is referred to as “uniform consistency in frequency domain”. Even though theoretically it provides much flexibility in choosing different sequences of values for t when estimating $\pi_{1,m}$, it does not have much practical value since t needs to be large for $\varphi_m(t, \boldsymbol{\mu})$ to converge to $\pi_{1,m}$ fast so that $\hat{\varphi}_m(t, \mathbf{z})$ can accurately estimate $\pi_{1,m}$.

Such uniform consistency is a consequence of the uniform boundedness and global Lipschitz property of the transform on $\{z_i\}_{i=1}^m$ that is used to construct $\{K(t, z_i; \mu_0)\}_{i=1}^m$. In contrast, for Constructions II and III, the corresponding transform is not necessarily uniformly bounded or globally Lipschitz (see the comparison below), and uniform consistency in frequency domain is hard to achieve. In fact, it is very challenging to derive good concentration inequalities for sums of transformed independent random variables where the transform is neither bounded nor globally Lipschitz. For progress along this line when the transform is a polynomial, we refer to readers to [Kim and Vu \(2000\)](#), [Vu \(2002\)](#) and [Schudy and Sviridenko \(2012\)](#).

Now we present the comparison. For location-shift families with RL type CFs, recall

$$K(t, x; \mu_0) = \int_{[-1,1]} \frac{\omega(s) w(ts, x)}{r_{\mu_0}(ts)} ds,$$

where $w(y, x) = \cos(yx - h_{\mu_0}(y))$ and $\|\partial_y h_{\mu_0}\|_{\infty} = C_{\mu_0} < \infty$ is assumed. So,

$$\|w\|_{\infty} < \infty \quad \text{and} \quad \|\partial_y w(\cdot, x)\|_{\infty} \leq \tilde{C}_0 |x| + \tilde{C} \tag{48}$$

for finite, positive constants \tilde{C}_0 and \tilde{C} that do not depend on x . This, together with the location-shift property, implies uniform consistency in frequency domain for $\hat{\varphi}_m(t, \mathbf{z})$ for an admissible ω . Let us examine Constructions II and III. First, consider Construction II. For (28), i.e.,

$$K(t, x; \theta_0) = H(\eta_0) \int_{[-1,1]} w(ts, x) \omega(s) ds \quad \text{with} \quad \eta_0 = e^{\theta_0},$$

where

$$w(y, x) = \frac{y^x \cos(2^{-1}\pi x - y\eta_0)}{c_x x!} \quad \text{for } y \geq 0 \text{ and } x \in \mathbb{N},$$

we have $\|w\|_\infty = \infty$, and $\|\partial_y w(\cdot, x)\|_\infty \leq \tilde{C}_0 |x| + \tilde{C}$ does not hold. Secondly, consider Construction III. Recall [Example 13](#) for Gamma family, i.e.,

$$K(t, x; \mu_0) = \Gamma(\sigma) \int_{[-1,1]} w(ts, x) \omega(s) ds,$$

where

$$w(y, x) = \sum_{n=0}^{\infty} \frac{(-yx)^n \cos(2^{-1}\pi n + y\xi(\theta_0))}{n! \Gamma(\sigma + n)} \quad \text{for } y \geq 0 \text{ and } x > 0.$$

Decompose $w(y, x)$ into the sum of four series

$$S_{l'}(x, y) = \sum_{l=0}^{\infty} \frac{(-yx)^{4l+l'} \cos(2^{-1}\pi(4l+l') + y\xi(\theta_0))}{(4l+l')! \Gamma(\sigma + 4l+l')} \quad \text{for } l' \in \{0, 1, 2, 3\}.$$

Then the summands in $S_{l'}(x, y)$ for each l' has a fixed sign uniformly in x and l . Further, there exists a sequence of $y \rightarrow \infty$ such that $|\cos(\frac{\pi}{2}n + y\xi(\theta_0))|$ is positive uniformly in n . Thus, there exists a sequence of x such that $\|w\|_\infty = \infty$.

C Proofs related to Construction I

C.1 Proof of [Theorem 1](#)

First of all, $K(t, x; \mu_0)$ defined by [\(10\)](#) is the real part of

$$K^\dagger(t, x; \mu_0) = \int_{[-1,1]} \frac{\omega(s) \exp(tsx)}{\hat{F}_{\mu_0}(ts)} ds$$

and $\psi(t, \mu; \mu_0)$ by [\(11\)](#) the real part of

$$\psi^\dagger(t, \mu; \mu_0) = \int_{[-1,1]} \omega(s) \frac{\hat{F}_\mu(ts)}{\hat{F}_{\mu_0}(ts)} ds = \int_{[-1,1]} \omega(s) \frac{r_\mu(ts)}{r_{\mu_0}(ts)} e^{t(h_\mu(ts) - h_{\mu_0}(ts))} ds. \quad (49)$$

With the boundedness of ω , the uniform continuity of r_μ for each $\mu \in U$, [\(7\)](#) and [\(8\)](#), we can apply Fubini theorem to obtain

$$\psi^\dagger(t, \mu; \mu_0) = \int K^\dagger(t, x; \mu_0) dF_\mu(x) = \int_{[-1,1]} \frac{\omega(s)}{\hat{F}_{\mu_0}(ts)} \nu(ds) \int \exp(tsx) dF_\mu(x).$$

This justifies [\(11\)](#). If $\mu = \mu_0$, then $\frac{\hat{F}_\mu}{\hat{F}_{\mu_0}} \equiv 1$, and [\(49\)](#) yields $\psi^\dagger(t, \mu_0; \mu_0) = 1$ since ω is a density on $[-1, 1]$, which justifies the first part of the second claim.

Finally, let $q_1(y) = \frac{r_\mu(y)}{r_{\mu_0}(y)}$ for $y \in \mathbb{R}$. Then (7) and (8) imply

$$\sup_{(s,t) \in [-1,1] \times \mathbb{R}} q_1(st) \leq C < \infty.$$

Since (9) holds, Theorem 3 of Costin et al. (2016) implies

$$\lim_{t \rightarrow \infty} \int_{[-1,1]} \omega(s) \frac{\hat{F}_\mu(ts)}{\hat{F}_{\mu_0}(ts)} ds = 0 \quad \text{for } \mu \neq \mu_0, \quad (50)$$

which justifies the third claim.

C.2 Proof of Corollary 1

It suffices to show (50). First of all, both $q_1(ts) = \frac{r_\mu(ts)}{r_{\mu_0}(ts)}$ and $\omega(s) q_1(ts)$ belong to $L^1([-1,1])$ uniformly in $t \in \mathbb{R}$. For any $\epsilon > 0$, there exists a step function $q_{1,\epsilon}$ on \mathbb{R} with compact support A_0 such that

$$q_{1,\epsilon}(y) = \sum_{j=1}^{n_{2,\epsilon}} a_{2j} 1_{A_{2j}}(y) \quad \text{and} \quad \int_{\mathbb{R}} |q_{1,\epsilon}(y) - q_1(y)| dy < \epsilon,$$

where $n_{2,\epsilon} \in \mathbb{N}$ is finite and the sets $\{A_{2j}\}_{j=1}^{n_{2,\epsilon}}$ are disjoint and $\bigcup_{j=1}^{n_{2,\epsilon}} A_{2j} \subseteq A_0$. Now consider t with $|t| \geq 1$. Then, the boundedness of ω and $\frac{r_\mu}{r_{\mu_0}}$ implies

$$\int_{[-1,1]} |\omega(s) q_1(ts) - \omega(s) q_{1,\epsilon}(ts)| ds \leq C |t|^{-1} \int_{\mathbb{R}} |q_{1,\epsilon}(y) - q_1(y)| dy \leq 2C\epsilon. \quad (51)$$

Let $\tau(ts) = h_\mu(ts) - h_{\mu_0}(ts)$ and

$$a_t^* = \max_{1 \leq j \leq n_{2,\epsilon}} \nu(\{s \in [-1,1] : ts \in A_{2j}\}).$$

Since the sets $\{A_{2,j}\}_{j=1}^{n_{2,\epsilon}}$ are uniformly bounded, $\lim_{|t| \rightarrow \infty} a_t^* = 0$ and

$$\int_{[-1,1]} |\omega(s) q_{1,\epsilon}(ts) \exp(\iota \tau(ts))| ds \leq C n_{2,\epsilon} a_t^* \rightarrow 0 \quad \text{as } |t| \rightarrow \infty. \quad (52)$$

Combining (51) and (52) gives (50), which justifies the claim.

C.3 Proof of Lemma 1

Since \mathcal{F} is a location-shift family, if z has CDF F_μ with $\mu \in U$, then there exists some $\mu_0 \in U$ such that $z = \mu' + z'$, where z' has CDF F_{μ_0} and $\mu' = \mu - \mu_0$. So,

$$\hat{F}_\mu(t) = \mathbb{E}[\exp(\iota tz)] = \mathbb{E}[\exp(\iota t(\mu' + z'))] = \hat{F}_{\mu_0}(t) \exp(\iota t \mu')$$

for all t . In particular, in the representation $\hat{F}_\mu = r_\mu e^{th_\mu}$, the modulus r_μ does not depend on μ and $h_\mu(t) = t\mu'$. If $\hat{F}_{\mu_0}(t) \neq 0$ for all $t \in \mathbb{R}$, then (9) holds and $\hat{\mathcal{F}}$ is of RL type.

C.4 Proof of Corollary 2

When \mathcal{F} is a location-shift family,

$$\int_A dF_\mu(x) = \int_{A-(\mu-\mu_0)} dF_{\mu_0}(y)$$

for each $A \subseteq \mathbb{R}$ measurable with respect to F_{μ_0} . Therefore,

$$\int K(t, x; \mu_0) dF_\mu(x) = \int K(t, y + (\mu - \mu_0); \mu_0) dF_{\mu_0}(y)$$

and the first identity in (12) holds. Further, $\frac{r_\mu}{r_{\mu_0}} \equiv 1$ for all $\mu \in U$. So, (11) reduces to the second identity in (12). Finally, since the cosine function is even on \mathbb{R} , it suffices to consider t and μ such that $t(\mu - \mu_0) > 0$ in the representation (12). When ω is good, the proof of the third claim of Lemma 7.1 of Jin (2006) remains valid, which implies $1 \geq \psi(t, \mu; \mu_0) \geq 0$ for all μ and t .

C.5 Proof of Theorem 2

Recall

$$K(t, x; \mu_0) = \int_{[-1,1]} \frac{\omega(s) \cos(tsx - h_{\mu_0}(ts))}{r_{\mu_0}(ts)} ds.$$

Set $w_i(y) = \cos(yz_i - h_{\mu_0}(y))$ for each i and $y \in \mathbb{R}$ and define

$$S_m(y) = \frac{1}{m} \sum_{i=1}^m (w_i(y) - \mathbb{E}[w_i(y)]). \quad (53)$$

Then

$$\begin{aligned} \hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\mu}) &= \frac{1}{m} \sum_{i=1}^m (K(t, z_i; \mu_0) - \mathbb{E}[K(t, z_i; \mu_0)]) \\ &= \int_{[-1,1]} \frac{\omega(s)}{r_{\mu_0}(ts)} S_m(ts) ds. \end{aligned}$$

Since $|w_i(ts)| \leq 1$ uniformly in (t, s, z_i, i) and $\{z_i\}_{i=1}^m$ are independent, (15) holds. Further, Hoeffding inequality of Hoeffding (1963) implies

$$\Pr \left(|S_m(ts)| \geq \frac{\lambda}{\sqrt{m}} \right) \leq 2 \exp(-2^{-1}\lambda^2) \text{ for any } \lambda > 0 \quad (54)$$

uniformly in $(t, s, m) \in \mathbb{R} \times [-1, 1] \times \mathbb{N}_+$. Recall $a(t; \mu_0) = \int_{[-1,1]} \frac{ds}{r_{\mu_0}(ts)}$ for $t \in \mathbb{R}$. Therefore,

$$|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\mu})| \leq \frac{\lambda \|\omega\|_\infty}{\sqrt{m}} \int_{[-1,1]} \frac{1}{r_{\mu_0}(ts)} ds = \frac{\lambda \|\omega\|_\infty}{\sqrt{m}} a(t; \mu_0)$$

with probability $1 - 2 \exp(-2^{-1} \lambda^2)$, i.e., (16) holds.

Consider the second claim. With probability at least $1 - 2 \exp(-2^{-1} \lambda_m^2)$,

$$\begin{aligned} \left| \frac{\hat{\varphi}_m(t_m, \mathbf{z})}{\pi_{1,m}} - 1 \right| &\leq \left| \frac{\hat{\varphi}_m(t_m, \mathbf{z}) - \varphi_m(t_m, \boldsymbol{\mu})}{\pi_{1,m}} \right| + \left| \frac{\varphi_m(t_m, \boldsymbol{\mu})}{\pi_{1,m}} - 1 \right| \\ &\leq \frac{1}{\pi_{1,m}} \frac{\lambda_m \|\omega\|_\infty}{\sqrt{m}} a(t_m; \mu_0) + \frac{1}{m\pi_{1,m}} \sum_{j \in I_{1,m}} |\psi(t_m, \mu_j; \mu_0)| \\ &\leq \frac{1}{\pi_{1,m}} \frac{\lambda_m \|\omega\|_\infty}{\sqrt{m}} a(t_m; \mu_0) + \sup_{(t, |\mu|) \in [t_m, \infty) \times [u_m, \infty)} |\psi(t, \mu; \mu_0)|. \end{aligned} \quad (55)$$

However, assumptions (17), (18) and $\{\mu_i : i \in I_{1,m}\} \subseteq [u_m, \infty)$ imply that both $\exp(-2^{-1} \lambda_m^2)$ and the upper bound in (55) converge to 0. So, (19) holds.

C.6 Proof of Corollary 3

Recall Theorem 2 and its proof. First of all, $\lim_{m \rightarrow \infty} \sup \{\psi(t_m, \mu_i; \mu_0) : i \in I_{1,m}\} = 0$ when $t_m \rightarrow \infty$ and $\min_{i \in I_{1,m}} |\mu_i - \mu_0| \geq (t_m)^{-1} \log \log m$. Therefore, it suffices to show

$$\lim_{m \rightarrow \infty} \frac{\lambda_m a(t_m; \mu_0)}{\pi_{1,m} \sqrt{m}} = 0 \text{ and } \lim_{m \rightarrow \infty} \exp(-2^{-1} \lambda_m^2) = 0, \quad (56)$$

where

$$a(t; \mu_0) = \int_{[-1,1]} \frac{ds}{r_{\mu_0}(ts)} \text{ for } t \in \mathbb{R}.$$

Consider $t > 0$. For Gaussian family, $r_{\mu_0}^{-1}(t) = \exp(2^{-1} t^2 \sigma^2)$. Setting $t_m = \sigma^{-1} \sqrt{2\gamma \log m}$ for any $\gamma \in (0, 0.5]$ gives

$$\int_{[0,1]} r_{\mu_0}^{-1}(ts) ds = \int_{[0,1]} \exp(2^{-1} t_m^2 s^2 \sigma^2) ds \leq \frac{m^\gamma}{\gamma \log m} (1 + o(1)), \quad (57)$$

where the last inequality follows from the proof of Theorem 4 of Jin (2008). Set $\lambda_m = o(t_m)$ with $\lambda_m \rightarrow \infty$. Then, (56) holds for all $\pi_{1,m} \geq Cm^{\gamma-0.5}$.

For Laplace family, $r_{\mu_0}^{-1}(t) = 1 + \sigma^2 t^2$ and

$$\int_{[0,1]} r_{\mu_0}^{-1}(ts) ds \leq \int_{[0,1]} (1 + \sigma^2 t^2) ds = 1 + \frac{\sigma^2 t^3}{3}. \quad (58)$$

Set $t_m = \log m$ and $\lambda_m = O(t_m)$ with $\lambda_m \rightarrow \infty$. Then, (56) holds for all $\pi_{1,m} \geq Cm^{-\gamma}$ with

$0 \leq \gamma < 1/2$.

For Hyperbolic Secant family, $r_\mu^{-1}(t) = \sigma \cosh(t\sigma^{-1}) \sim 2^{-1}\sigma \exp(\sigma^{-1}t)$ as $t \rightarrow \infty$ and

$$\int_{[0,1]} r_\mu^{-1}(ts) ds \leq C\sigma \int_{[0,1]} \exp(\sigma^{-1}ts) ds \leq C\sigma^2 \frac{\exp(\sigma^{-1}t)}{t}. \quad (59)$$

Set $t_m = \sigma\gamma \log m$ with $0 < \gamma \leq 1/2$ and $\lambda_m = o(t_m)$ with $\lambda_m \rightarrow \infty$. Then, (56) holds for all $\pi_{1,m} \geq Cm^{\gamma-0.5}$.

For Logistic family, $r_\mu^{-1}(t) = (\pi\sigma t)^{-1} \sinh(\pi\sigma t) \sim (2\pi\sigma t)^{-1} e^{\pi\sigma t}$ as $t \rightarrow \infty$ and

$$\int_{[0,1]} r_\mu^{-1}(t) ds \leq (\sigma\pi t)^{-1} \int_{[0,1]} \exp(\sigma\pi ts) ds = \frac{\exp(\sigma\pi t)}{\sigma^2\pi^2 t^2}. \quad (60)$$

Set $t_m = (\sigma\pi)^{-1} \gamma \log m$ with $0 < \gamma \leq 1/2$ and $\lambda_m = o(t_m)$ with $\lambda_m \rightarrow \infty$. Then, (56) holds for all $\pi_{1,m} \geq Cm^{\gamma-0.5}$.

For Cauchy family, $r_\mu^{-1}(t) = \exp(\sigma|t|)$ and

$$\int_{[0,1]} r_\mu^{-1}(ts) ds \leq \int_{[0,1]} \exp(\sigma|t|s) ds \leq \frac{\exp(\sigma|t|)}{\sigma|t|}. \quad (61)$$

Set $t_m = \sigma^{-1}\gamma \log m$ with $0 < \gamma \leq 1/2$ and $\lambda_m = o(t_m)$ with $\lambda_m \rightarrow \infty$. Then, (56) holds for all $\pi_{1,m} \geq Cm^{\gamma-0.5}$.

In each case above, $C > 0$ can be any constant for which $\pi_{1,m} \in (0, 1]$ as γ varies in its designated range.

C.7 Proof of Theorem 3

The strategy of proof adapts that for Lemma 7.2 of Jin (2006) for Gaussian family, which can be regarded as an application of the ‘‘chaining method’’ proposed by Talagrand (1996). Since r_{μ_0} has no real zeros and \mathcal{F} is a location-shift family, r_μ has no real zeros for each $\mu \neq \mu_0$ and $h_\mu(t)$ is well-defined and continuous in t on \mathbb{R} for each $\mu \in U$. Therefore, $\frac{d}{dy} h_{\mu_0}(y)$ can be defined.

Recall $w_i(y) = \cos(yz_i - h_{\mu_0}(y))$ and $S_m(y)$ defined by (53). Let $\hat{s}_m(y) = \frac{1}{m} \sum_{i=1}^m w_i(y)$ and $s_m(y) = \mathbb{E}[\hat{s}_m(y)]$. For the rest of the proof, we will first assume the existence of the positive constants γ' , γ'' , q , ϑ and the non-negative constant ϑ' and then determine them at the end of the proof. Let $\gamma_m = \gamma' \log m$. The rest of the proof is divided into three parts.

Part I: to show the assertion ‘‘if

$$\lim_{m \rightarrow \infty} \frac{m^\vartheta \log \gamma_m}{R_m(\rho) \sqrt{m} \sqrt{2q\gamma_m}} = \infty \quad (62)$$

where $R_m(\rho) = \mathbb{E}[|X_1|] + 2\rho + 2C_{\mu_0}$ and X_1 has CDF F_{μ_0} , then, for all large m ,

$$\sup_{\mu \in \mathcal{B}_m(\rho)} \sup_{y \in [0, \gamma_m]} |\hat{s}_m(y) - \mathbb{E}[\hat{s}_m(y)]| \leq \frac{\sqrt{2q\gamma_m}}{\sqrt{m}} \quad (63)$$

holds with probability at least $1 - p_m(\vartheta, q, h_{\mu_0}, \gamma_m)$, where

$$p_m(\vartheta, q, h_{\mu_0}, \gamma_m) = 2m^\vartheta \gamma_m^2 \exp(-q\gamma_m) + 4A_{\mu_0} q \gamma_m m^{-2\vartheta} (\log \gamma_m)^{-2} \quad (64)$$

and A_{μ_0} is the variance of $|X_1|$.

Define the closed interval $G_m = [0, \gamma_m]$. Let $\mathcal{P} = \{t_1, \dots, t_{l_*}\}$ for some $l_* \in \mathbb{N}_+$ with $t_j < t_{j+1}$ be a partition of G_m with norm $\Delta = \max_{1 \leq j \leq l_*-1} |t_{j+1} - t_j|$ such that $\Delta = m^{-\vartheta}$. For each $y \in G_m$, pick $t_j \in \mathcal{P}$ that is the closest to y . By Lagrange mean value theorem,

$$\begin{aligned} |\hat{s}_m(y) - s_m(y)| &\leq |\hat{s}_m(y_i) - s_m(y_i)| + |(\hat{s}_m(y) - \hat{s}_m(y_i)) - (s_m(y) - s_m(y_i))| \\ &\leq |\hat{s}_m(y_i) - s_m(y_i)| + \Delta \sup_{y \in \mathbb{R}} |\partial_y (\hat{s}_m(y) - s_m(y))|, \end{aligned}$$

where ∂ denotes the derivative with respect to the subscript. So,

$$B_0 = \Pr \left(\sup_{\mu \in \mathcal{B}_m(\rho)} \max_{y \in G_m} |S_m(y)| \geq \frac{\sqrt{2q\gamma_m}}{\sqrt{m}} \right) \leq B_1 + B_2, \quad (65)$$

where

$$B_1 = \Pr \left(\sup_{\mu \in \mathcal{B}_m(\rho)} \max_{1 \leq i \leq l_*} |\hat{s}_m(y_i) - s_m(y_i)| \geq \frac{\sqrt{2q\gamma_m} - (2q\gamma_m)^{-1/2} \log \gamma_m}{\sqrt{m}} \right)$$

and

$$B_2 = \Pr \left(\sup_{\mu \in \mathcal{B}_m(\rho)} \sup_{y \in \mathbb{R}} |\partial_y (\hat{s}_m(y) - s_m(y))| \geq \frac{\Delta^{-1} (2q\gamma_m)^{-1/2} \log \gamma_m}{\sqrt{m}} \right).$$

Applying to B_1 the union bound and Hoeffding inequality (54) gives

$$B_1 \leq 2l_* \exp(-q\gamma_m + \log \gamma_m) \exp \left(-\frac{(\log \gamma_m)^2}{4q\gamma_m} \right) \leq 2m^\vartheta \gamma_m^2 \exp(-q\gamma_m). \quad (66)$$

On the other hand, $\partial_y w_i(y) = -(z_i - \partial_y h_{\mu_0}(y)) \sin(yz_i - h_{\mu_0}(y))$, and

$$\partial_y \mathbb{E}[w_i(y)] = \mathbb{E}[\partial_y w_i(y)] = -\mathbb{E}[(z_i - \partial_y h_{\mu_0}(y)) \sin(yz_i - h_{\mu_0}(y))]$$

holds since $\int |x|^2 dF_\mu(x) < \infty$ for each $\mu \in U$ and $\sup_{y \in \mathbb{R}} \left| \frac{d}{dy} h_{\mu_0}(y) \right| = C_{\mu_0} < \infty$. So,

$$\sup_{y \in \mathbb{R}} |\partial_y (\hat{s}_m(y) - s_m(y))| \leq \frac{1}{m} \sum_{i=1}^m |z_i| + 2C_{\mu_0} + \frac{1}{m} \sum_{i=1}^m \mathbb{E}[|z_i|]. \quad (67)$$

Since \mathcal{F} is a location-shift family, there are independent and identically distributed (i.i.d.) $\{X_i\}_{i=1}^m$ with common CDF F_{μ_0} such that $z_i = (\mu_i - \mu_0) + X_i$ for $1 \leq i \leq m$. Therefore, the upper bound in (67) satisfies

$$\begin{aligned} & \frac{1}{m} \sum_{i=1}^m |z_i| + 2C_{\mu_0} + \frac{1}{m} \sum_{i=1}^m \mathbb{E}[|z_i|] \\ & \leq \frac{1}{m} \sum_{i=1}^m |X_i| + \frac{2}{m} \sum_{i=1}^m |\mu_i - \mu_0| + 2C_{\mu_0} + \frac{1}{m} \sum_{i=1}^m \mathbb{E}[|X_i|] \\ & \leq \frac{1}{m} \sum_{i=1}^m (|X_i| - \mathbb{E}[|X_i|]) + R_m(\rho), \end{aligned}$$

where we recall $R_m(\rho) = \mathbb{E}[|X_1|] + 2\rho + 2C_{\mu_0}$. Namely,

$$\sup_{y \in \mathbb{R}} |\partial_y (\hat{s}_m(y) - s_m(y))| \leq \frac{1}{m} \sum_{i=1}^m (|X_i| - \mathbb{E}[|X_i|]) + R_m(\rho)$$

and

$$\sup_{\mu \in \mathcal{B}_m(\rho)} \sup_{y \in \mathbb{R}} |\partial_y (\hat{s}_m(y) - s_m(y))| \leq \frac{1}{m} \sum_{i=1}^m (|X_i| - \mathbb{E}[|X_i|]) + R_m(\rho).$$

When (62) holds, Chebyshev inequality implies

$$\begin{aligned} B_{2,1} &= \Pr \left(\frac{1}{m} \sum_{i=1}^m (|X_i| - \mathbb{E}[|X_i|]) \geq \frac{\Delta^{-1} (2q\gamma_m)^{-1/2} \log \gamma_m}{\sqrt{m}} - R_m(\rho) \right) \\ &\leq 4A_{\mu_0} q \gamma_m m^{-2\vartheta} (\log \gamma_m)^{-2} \end{aligned}$$

for all m large enough, where A_{μ_0} is the variance of $|X_1|$. Thus, for all m large enough,

$$B_2 \leq B_{2,1} \leq 4A_{\mu_0} q \gamma_m m^{-2\vartheta} (\log \gamma_m)^{-2}.$$

This, together with (66) and (65) and the continuity of $\hat{s}_m(y) - s_m(y)$ in y , implies

$$B_0 = \Pr \left(\sup_{\mu \in \mathcal{B}_m(\rho)} \sup_{y \in \tilde{G}_m} |\hat{s}_m(y) - s_m(y)| \geq \frac{\sqrt{2q\gamma_m}}{\sqrt{m}} \right) \leq p_m(\vartheta, q, h_{\mu_0}, \gamma_m)$$

for all m large enough. This justifies the assertion.

Part II: to show the uniform bound on $|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\mu})|$. Pick a positive sequence

$\{\tau_m : m \geq 1\}$ such that $\tau_m \leq \gamma_m$ for all large m and $\tau_m \rightarrow \infty$. Then, **Part I** implies that, with probability at least $1 - p_m(\vartheta, q, h_{\mu_0}, \gamma_m)$,

$$\begin{aligned} & \sup_{\boldsymbol{\mu} \in \mathcal{B}_m(\rho)} \sup_{t \in [0, \tau_m]} |\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\mu})| \\ & \leq \sup_{\boldsymbol{\mu} \in \mathcal{B}_m(\rho)} \sup_{t \in [0, \tau_m]} \int_{[-1, 1]} \omega(s) \frac{\sup_{t \in G_m} |S_m(ts)|}{r_{\mu_0}(ts)} ds \leq \Upsilon(q, \tau_m, \gamma_m, r_{\mu_0}) \end{aligned}$$

for all sufficiently large m , where

$$\Upsilon(q, \tau_m, \gamma_m, r_{\mu_0}) = \frac{2 \|\omega\|_{\infty} \sqrt{2q\gamma_m}}{\sqrt{m}} \sup_{t \in [0, \tau_m]} \int_{[0, 1]} \frac{ds}{r_{\mu_0}(ts)}.$$

Part III: to determine the constants γ' , γ'' , q , ϑ and ϑ' and a uniform consistency class. Set γ' , ϑ and q such that $q\gamma' > \vartheta > 2^{-1}$ and $0 \leq \vartheta' < \vartheta - 1/2$. Then $p_m(\vartheta, q, h_{\mu_0}, \gamma_m) \rightarrow 0$, $B_0 \rightarrow 0$ and $m^{\vartheta-1/2} \gamma_m^{-1/2} \log \gamma_m \rightarrow \infty$ as $m \rightarrow \infty$. If additionally $R_m(\rho) = O(m^{\vartheta'})$ and $u_m \geq (\gamma'' \tau_m)^{-1} \log \log m$. Then $\tau_m u_m \rightarrow \infty$ as $m \rightarrow \infty$ and (62) holds.

Recall

$$\psi(t_m, \mu_i; \mu_0) = \int_{[-1, 1]} \omega(s) \cos(t_m s (\mu_i - \mu_0)) ds.$$

Since $\gamma_m |\mu_i - \mu_0| \geq \gamma_m |u_m - \mu_0| \rightarrow \infty$ uniformly for $i \in I_{1,m}$,

$$\lim_{m \rightarrow \infty} \sup \{\psi(t, \mu; \mu_0) : (t, |\mu|) \in [\gamma_m, \infty) \times [u_m, \infty)\} = 0.$$

So, when $\pi_{1,m}^{-1} \Upsilon(q, \tau_m, \gamma_m, r_{\mu_0}) \rightarrow 0$, the same reasoning used to prove (19) implies

$$\Pr \left(\sup_{\boldsymbol{\mu} \in \mathcal{B}_m(\rho)} \left| \pi_{1,m}^{-1} \sup_{t \in [0, \tau_m]} \hat{\varphi}_m(t, \mathbf{z}) - 1 \right| \rightarrow 0 \right) \rightarrow 1.$$

In other words, as claimed,

$$\mathcal{Q}_m(\boldsymbol{\mu}, t; \mathcal{F}) = \left\{ \begin{array}{l} q\gamma' > \vartheta > 2^{-1}, \gamma' > 0, \gamma'' > 0, 0 \leq \vartheta' < \vartheta - 1/2, \\ R_m(\rho) = O(m^{\vartheta'}), \tau_m \leq \gamma_m, u_m \geq \frac{\log \log m}{\gamma'' \tau_m}, \\ t \in [0, \tau_m], \lim_{m \rightarrow \infty} \pi_{1,m}^{-1} \Upsilon(q, \tau_m, \gamma_m, r_{\mu_0}) = 0 \end{array} \right\}$$

is a uniform consistency class.

C.8 Proof of Corollary 4

For the proof, we will refer to the proofs of **Theorem 3** and **Corollary 3**. For Gaussian family, when $q\sigma^{-1} > \vartheta > 2^{-1}$, we can set $\gamma_m = \sigma^{-1} \log m$ and $\tau_m = \sigma^{-1} \sqrt{2\gamma \log m}$. Then (57) implies the claimed uniform consistency class. Further, we see that the fastest speed of convergence is $\sqrt{\log m}$, achieved when $\liminf_{m \rightarrow \infty} \pi_{1,m} > 0$.

For Laplace family, $r_{\mu_0}^{-1}(t) = 1 + \sigma^2 t^2$,

$$\sup_{t \in [0, \gamma_m]} \int_{[0,1]} r_{\mu_0}^{-1}(ts) ds = \int_{[0,1]} (1 + \sigma^2 \gamma_m^2 s^2) ds = 1 + \frac{\sigma^2 \gamma_m^2}{3},$$

and

$$\Upsilon(q, \gamma_m, \gamma_m, r_{\mu_0}) = \frac{2 \|\omega\|_{\infty} \sqrt{2q\gamma_m}}{\sqrt{m}} \left(1 + \frac{\sigma^2 \gamma_m^2}{3}\right).$$

So, setting $\tau_m = \gamma_m = \log m$ and $0 \leq \gamma < 1/2$ gives the claimed uniform consistency class.

For Hyperbolic Secant family, $r_{\mu}^{-1}(t) = \sigma \cosh(t\sigma^{-1}) \sim 2^{-1} \sigma \exp(\sigma^{-1}t)$ as $t \rightarrow \infty$ and

$$\sup_{t \in [0, \gamma_m]} \int_{[0,1]} r_{\mu}^{-1}(ts) ds \leq C \sup_{t \in [0, \gamma_m]} \int_{[0,1]} \exp(\sigma^{-1}ts) ds \leq C \exp(\sigma^{-1}\gamma_m)$$

and

$$\Upsilon(q, \gamma_m, \gamma_m, r_{\mu_0}) \leq \frac{C \sqrt{2q\gamma_m}}{\sqrt{m}} \exp(\sigma^{-1}\gamma_m).$$

When $q\sigma > \vartheta > 2^{-1}$, we can set $\gamma_m = \sigma \log m$ and $\tau_m = \sigma \gamma \log m$ with $0 < \gamma < 1/2$. This gives the claimed uniform consistency class.

For Logistic family, $r_{\mu}^{-1}(t) = (\pi\sigma t)^{-1} \sinh(\pi\sigma t) \sim (2\pi\sigma t)^{-1} e^{\pi\sigma t}$ as $t \rightarrow \infty$. Fix a small $\varepsilon'' \in (0, 1)$. We can pick a small $\varepsilon' \in (0, 1)$ such that $r_{\mu}^{-1}(t) \geq 1 - \varepsilon''$ for all $t \in [0, \varepsilon']$. Then

$$\sup_{t \in [0, \gamma_m]} \int_{[0,1]} r_{\mu}^{-1}(ts) ds \leq C \left(1 + \frac{1}{\varepsilon'} \sup_{t \in [0, \gamma_m]} \int_{[0,1]} \exp(\sigma\pi ts) ds\right) \leq C \exp(\sigma\pi\gamma_m).$$

So,

$$\Upsilon(q, \gamma_m, \gamma_m, r_{\mu_0}) \leq \frac{C \sqrt{\gamma_m}}{\sqrt{m}} \exp(\sigma\pi\gamma_m).$$

When $q(\sigma\pi)^{-1} > \vartheta > 2^{-1}$, we can set $\gamma_m = (\sigma\pi)^{-1} \log m$ and $\tau_m = (\sigma\pi)^{-1} \gamma \log m$ with $0 < \gamma < 1/2$. This then gives the claimed uniform consistency class.

D Proofs related to Construction II

D.1 Proof of **Theorem 4**

Clearly, $L(\theta) = H(e^{\theta})$ and $c_k = \frac{H^{(k)}(0)}{k!}$, where $H^{(k)}$ is the k th order derivative of H and $H^{(0)} = H$. Let

$$K^{\dagger}(t, x; \theta_0) = H(e^{\theta_0}) \int_{[-1,1]} \frac{(ts)^x}{\exp(tse^{\theta_0}) H^{(x)}(0)} \omega(s) ds \quad (68)$$

and $\psi^\dagger(t, \theta; \theta_0) = \int K^\dagger(t, x; \theta_0) dG_\theta(x)$. Then

$$\begin{aligned}\psi^\dagger(t, \theta; \theta_0) &= \int K^\dagger(t, x; \theta_0) dG_\theta(x) \\ &= \frac{H(e^{\theta_0})}{H(e^\theta)} \int_{[-1,1]} \sum_{k=0}^{\infty} \frac{(t s)^k e^{\theta k} c_k}{\exp(t s e^{\theta_0}) H^{(k)}(0)} \omega(s) ds \\ &= \frac{H(e^{\theta_0})}{H(e^\theta)} \int_{[-1,1]} \exp\left(\iota s t (e^\theta - e^{\theta_0})\right) \omega(s) ds,\end{aligned}$$

for which $\psi^\dagger(t, \theta_0; \theta_0) = 1$ for any t and $\lim_{t \rightarrow \infty} \psi^\dagger(t, \theta; \theta_0) = 0$ for each $\theta \neq \theta_0$ by the RL Lemma. Taking the real parts of K^\dagger and ψ^\dagger yields the claim.

D.2 Proof of Lemma 2

By simple calculations, we obtain the following: (1) $c_k k! \equiv 1$ for Poisson family; (2) $(c_k k!)^{-1} = \frac{(n-1)!}{(k+n-1)!}$ for Negative Binomial family with a fixed n ; (3) $(c_k k!)^{-1} = (1+k)^{-(k-1)}$ for Abel family; (4) $(c_k k!)^{-1} = (k+1)!((2k)!)^{-1}$ for Takács family. Therefore, (40) holds. Fix a $\sigma > 0$. Then for Strict Arcsine family,

$$c_k k! = c_k^*(1) \geq 2^{k-2} \left(([2^{-1}k] - 1)! \right)^2,$$

and for Large Arcsine family,

$$c_k k! = \frac{k+1}{c_k^*(1+k)} \geq (1+k)^{[2^{-1}k]-1} 2^{k-2} \left(([2^{-1}k] - 1)! \right)^2.$$

So, (40) does not hold for these two families.

Now we show the third claim. Since $H(z) = \sum_{k=0}^{\infty} c_k z^k$ has a positive radius R_H of convergence, there exists $R_H > \tilde{r} > 0$ such that

$$H^{(k)}(0) = \frac{k!}{2\pi \iota} \int_{\{z \in \mathbb{C}; |z|=\tilde{r}\}} \frac{H(z)}{z^{k+1}} dz \quad \text{for all } k \in \mathbb{N}.$$

However, H has all positive coefficients. Therefore, $\max_{\{z \in \mathbb{C}; |z|=\tilde{r}\}} |H(z)|$ is achieved when $z = \tilde{r}$, and

$$\left| H^{(k)}(0) \right| \leq \frac{k!}{2\pi} 2\pi \tilde{r} \frac{\sup_{|z|=\tilde{r}} |H(z)|}{\tilde{r}^{k+1}} = H(\tilde{r}) \frac{k!}{\tilde{r}^k}.$$

Observing that $H^{(k)}(0)$ is real and $H^{(k)}(0) = c_k k!$ for $k \in \mathbb{N}$ gives (41).

D.3 Proof of Theorem 6

First, we prove the following lemma.

Lemma 3 *If Z has CDF G_θ with GF H , (40) holds and $\eta > 0$, then*

$$\mathbb{E} \left[t^{2Z} \left(H^{(Z)}(0) \right)^{-2} \right] \leq \frac{C}{L(\theta)} \frac{\exp(2t\sqrt{\eta})}{\sqrt{t\sqrt{\eta}}} \quad (69)$$

for positive and sufficiently large t .

Proof. Recall $H(z) = \sum_{k=0}^{\infty} c_k z^k$ and $H^{(k)}(0) = c_k k!$ for $k \in \mathbb{N}$. Let $\chi_Z(t) = \mathbb{E} \left[t^{2Z} \left(H^{(Z)}(0) \right)^{-2} \right]$. Since (40) holds,

$$\chi_Z(t) = \frac{1}{L(\theta)} \sum_{k=0}^{\infty} \frac{t^{2k} c_k \eta^k}{(c_k k!)^2} \leq \frac{C}{L(\theta)} B_{\text{II}}(2t\sqrt{\eta}),$$

where $B_{\text{II}}(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k!)^2}$ for $x > 0$. So, it suffices to bound $B_{\text{II}}(x)$. For $\sigma > 0$ and $y \in \mathbb{C}$, let

$$J_\sigma(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + \sigma + 1)} \left(\frac{y}{2} \right)^{2n + \sigma} \quad (70)$$

Then J_σ is the Bessel function of the first kind of order σ ; see definition (1.17.1) in Chapter 1 of Szegő (1975), and $B_{\text{II}}(x) = J_0(-ix)$. By identity (1.71.8) in Chapter 1 of Szegő (1975) that was derived on page 368 of Whittaker and Watson (1940), we have

$$J_\sigma(y) = \sqrt{2} (\pi y)^{-1/2} \cos(y - c_0) (1 + O(y^{-2})) \quad (71)$$

as $y \rightarrow \infty$ whenever $|\arg y| < \pi$, where $c_0 = 2^{-1}\sigma\pi - 4^{-1}\pi$. So,

$$B_{\text{II}}(2t\sqrt{\eta}) = |J_0(-2it\sqrt{\eta})| = C (t\sqrt{\eta})^{-1/2} \exp(2t\sqrt{\eta}) \quad (72)$$

as $0 < t\eta \rightarrow \infty$, where we have used the identity $|\cos z|^2 = \cosh^2(\Im(z)) - \sin^2(\Re(z))$. The bound given by (72) is tight up to a multiple of a positive constant, which can be seen from inequality (5) of Gronwall (1932). On the other hand, $B_{\text{II}}(t\eta) = O(1)$ when $t\eta = O(1)$. Thus, when $\eta > 0$ and (40) holds, (69) holds for all positive and sufficiently large t . ■

Now we show the first claim of the theorem. Recall $\eta_i = e^{\theta_i}$ when z_i has CDF G_{θ_i} for $1 \leq i \leq m$. Let $V_m(\hat{\varphi}) = \mathbb{V}[\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})]$ and $\tilde{p}_m(\lambda) = \Pr(|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})| \geq \lambda)$. Let

$$w(t, x) = \frac{t^x \cos\left(\frac{\pi}{2}x - t\eta_0\right)}{c_x x!} \quad \text{for } t \geq 0 \text{ and } x \in \mathbb{N} \quad (73)$$

and

$$S_m(t) = \frac{1}{m} \sum_{i=1}^m (w(t, z_i) - \mathbb{E}[w(t, z_i)]) \quad \text{for } t \geq 0.$$

Then (28) is equivalent to $K(t, x; \theta_0) = H(\eta_0) \int_{[-1,1]} w(ts, x) \omega(s) ds$ and

$$\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta}) = H(\eta_0) \int_{[-1,1]} S_m(ts) \omega(s) ds.$$

Since $\mathbb{E} [w^2 (t, Z)] \leq \chi_Z (t)$, inequality (69) implies

$$V_m (\hat{\varphi}) \leq \frac{C}{m^2} \sum_{i=1}^m \frac{\exp (2t \sqrt{\eta_i})}{L (\theta_i) (t \sqrt{\eta_i})^{1/2}} \leq \frac{C}{m} \frac{\exp (2t \|\boldsymbol{\eta}\|_\infty^{1/2})}{\sqrt{t} \phi_m (L, \boldsymbol{\theta})}.$$

So,

$$\Pr (|S_m (t)| \geq \lambda) \leq \frac{C V_m (\hat{\varphi})}{\lambda^2} \leq \frac{C}{\lambda^2 m} \frac{\exp (2t \|\boldsymbol{\eta}\|_\infty^{1/2})}{\sqrt{t} \phi_m (L, \boldsymbol{\theta})}.$$

From

$$\int_{[-1,1]} S_m (ts) \omega (s) ds = \frac{1}{t} \int_{[-t,t]} S_m (y) \omega (yt^{-1}) dy$$

for $t > 0$, we have

$$\tilde{p}_m (\lambda) \leq \Pr \left(|S_m (t)| \geq \frac{\lambda}{2H (\eta_0) \|\omega\|_\infty} \right) \leq \frac{C H^2 (\eta_0) \|\omega\|_\infty^2}{\lambda^2 m} \frac{\exp (2t \|\boldsymbol{\eta}\|_\infty^{1/2})}{\sqrt{t} \phi_m (L, \boldsymbol{\theta})}.$$

Finally, we show the second claim of the theorem. For Poisson family, $c_k k! = 1$ for all $k \in \mathbb{N}$. So, from (73) we obtain

$$\mathbb{E} [w^2 (t, Z)] \leq \frac{H^2 (\eta_0) \exp (t^2 \sqrt{\eta})}{L (\theta)} \quad \text{and} \quad V_m (\hat{\varphi}) \leq \frac{C}{m} \frac{\exp (t^2 \|\boldsymbol{\eta}\|_\infty^{1/2})}{\min_{1 \leq i \leq m} L (\theta_i)}$$

and

$$\tilde{p}_m (\lambda) \leq \Pr \left(|S_m (t)| \geq \frac{\lambda}{2H (\eta_0) \|\omega\|_\infty} \right) \leq \frac{C}{\lambda^2 m} \frac{\exp (t^2 \|\boldsymbol{\eta}\|_\infty^{1/2})}{\min_{1 \leq i \leq m} L (\theta_i)}$$

for positive and sufficiently large t .

D.4 Proof of Theorem 7

Obviously, $\phi_m (L, \boldsymbol{\theta})$ is positive and finite when $\|\boldsymbol{\theta}\|_\infty \leq \rho$. First, consider the case when (40) holds. Then (42) implies

$$\Pr (|\hat{\varphi}_m (t, \mathbf{z}) - \varphi_m (t, \boldsymbol{\theta})| \geq \lambda) \leq \frac{C}{\lambda^2 m} \frac{\exp (2t \|\boldsymbol{\eta}\|_\infty^{1/2})}{\sqrt{t}}.$$

We can set $t = 2^{-1} \|\boldsymbol{\eta}\|_\infty^{-1/2} \gamma \log m$ for $\gamma \in (0, 1]$, which induces

$$\Pr (|\hat{\varphi}_m (t, \mathbf{z}) - \varphi_m (t, \boldsymbol{\theta})| \geq \lambda) \leq \frac{C}{\lambda^2 m^{1-\gamma}} \frac{1}{\sqrt{\gamma \log m}}.$$

Let $\varepsilon > 0$ be any finite constant. If $\pi_{1,m} \geq Cm^{(\gamma-1)/2}$, then

$$\Pr\left(\frac{|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})|}{\pi_{1,m}} \geq \varepsilon\right) \leq \frac{C\varepsilon^{-2}}{\sqrt{\gamma \log m}} \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Moreover,

$$\psi(t, \theta; \theta_0) = \frac{H(\eta_0)}{H(\eta)} \int_{[-1,1]} \cos(st(\eta - \eta_0)) \omega(s) ds \rightarrow 0 \text{ as } m \rightarrow \infty$$

whenever $\lim_{m \rightarrow \infty} t \min_{1 \leq i \leq m} |\eta_0 - \eta_i| = \infty$.

Secondly, we deal with Poisson family. Clearly, $L_{\min}^{(m)} = \min_{1 \leq i \leq m} L(\theta_i)$ is positive and finite when $\|\boldsymbol{\theta}\|_\infty \leq \rho$. So, inequality (43) implies

$$\Pr(|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})| \geq \lambda) \leq \frac{C}{\lambda^2 m} \exp\left(t^2 \|\boldsymbol{\eta}\|_\infty^{1/2}\right).$$

So, we can set $t = \sqrt{\|\boldsymbol{\eta}\|_\infty^{-1/2} \gamma \log m}$ for a fixed $\gamma \in (0, 1)$, which induces

$$\Pr(|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})| \geq \lambda) \leq \frac{C}{\lambda^2 m^{1-\gamma}}.$$

If $\pi_{1,m} \geq Cm^{(\gamma'-1)/2}$ for any $\gamma' > \gamma$, then

$$\Pr\left(\frac{|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})|}{\pi_{1,m}} \geq \varepsilon\right) \leq \frac{C\varepsilon^{-2}}{m^{\gamma'-\gamma}} \rightarrow 0 \text{ as } m \rightarrow \infty.$$

This completes the proof.

E Proof related to Construction III

E.1 Proof of Theorem 5

Let

$$K^\dagger(t, x; \mu_0) = \int_{[-1,1]} \exp(its\xi(\theta_0)) \sum_{n=0}^{\infty} \frac{(-tsx)^n}{\tilde{a}_n n!} \omega(s) ds. \quad (74)$$

Since $\{\tilde{c}_n(\theta)\}_{n \geq 1}$ is separable at θ_0 , then

$$\begin{aligned}
\psi^\dagger(t, \mu; \mu_0) &= \frac{1}{\zeta(\theta_0)} \int K^\dagger(t, x; \theta_0) dG_\theta(x) \\
&= \frac{1}{\zeta(\theta_0)} \int dG_\theta(x) \int_{[-1,1]} \exp(its\xi(\theta_0)) \sum_{n=0}^{\infty} \frac{(-tsx)^n}{\tilde{a}_n n!} \omega(s) ds \\
&= \frac{1}{\zeta(\theta_0)} \int_{[-1,1]} \exp(its\xi(\theta_0)) \omega(s) ds \sum_{n=0}^{\infty} \frac{(-its)^n}{\tilde{a}_n n!} \tilde{c}_n(\theta) \\
&= \frac{\zeta(\theta)}{\zeta(\theta_0)} \int_{[-1,1]} \exp(its\xi(\theta_0)) \omega(s) ds \sum_{n=0}^{\infty} \frac{(-its)^n}{n!} \xi^n(\theta) \\
&= \frac{\zeta(\theta)}{\zeta(\theta_0)} \int_{[-1,1]} \exp(its(\xi(\theta_0) - \xi(\theta))) \omega(s) ds. \tag{75}
\end{aligned}$$

Further, $\psi^\dagger(t, \mu; \mu_0) = 1$ when $\mu = \mu_0$ for all t , and the RL Lemma implies that $\lim_{t \rightarrow \infty} \psi^\dagger(t, \mu; \mu_0) = 0$ for each $\theta \neq \theta_0$. Taking the real parts of K^\dagger and ψ^\dagger gives the claim.

E.2 Proof of Theorem 8

First, we prove the following lemma.

Lemma 4 For a fixed $\sigma > 0$, let

$$\tilde{w}(z, x) = \sum_{n=0}^{\infty} \frac{(zx)^n}{n! \Gamma(\sigma + n)} \quad \text{for } z, x > 0.$$

If Z has CDF G_θ from the Gamma family with scale parameter σ , then

$$\mathbb{E}[\tilde{w}^2(z, Z)] \leq C \left(\frac{z}{1-\theta} \right)^{3/4-\sigma} \exp\left(\frac{4z}{1-\theta} \right) \tag{76}$$

for positive and sufficiently large z .

Proof. Recall the Bessel function J_σ defined by (70) and the asymptotic bound (71). Then,

$$\tilde{w}(z, x) = (\sqrt{zx})^{1-\sigma} |t^{1-\sigma} J_{\sigma-1}(t2\sqrt{zx})| = (zx)^{\frac{1}{4}-\frac{\sigma}{2}} \exp(2\sqrt{zx}) \left(1 + O\left((zx)^{-1} \right) \right)$$

when $zx \rightarrow \infty$. Let $A_{1,z} = \{x \in (0, \infty) : zx = O(1)\}$. Then, on the set $A_{1,z}$, $f_\theta(x) = O(x^{\sigma-1})$ and $\tilde{w}(z, x) \leq Ce^{zx} = O(1)$ when $\theta < 1$. Therefore,

$$\int_{A_{1,z}} \tilde{w}^2(z, x) dG_\theta(x) \leq C(1-\theta)^\sigma \int_{A_{1,z}} x^{\sigma-1} dx \leq C(1-\theta)^\sigma z^{-\sigma}. \tag{77}$$

On the other hand, let $A_{2,z} = \{x \in (0, \infty) : \lim_{z \rightarrow \infty} zx = \infty\}$. Then

$$\begin{aligned} \int_{A_{2,z}} \tilde{w}^2(z, x) dG_\theta(x) &\leq C \int_{A_{2,z}} (zx)^{\frac{1}{2}-\sigma} \exp(4\sqrt{zx}) dG_\theta(x) \\ &\leq \int_{A_{2,z}} (zx)^{\frac{1}{2}-\sigma} \sum_{n=0}^{\infty} \frac{(4\sqrt{zx})^n}{n!} dG_\theta(x) = z^{\frac{1}{2}-\sigma} B_{\text{III}}(z), \end{aligned} \quad (78)$$

where

$$B_{\text{III}}(z) = \sum_{n=0}^{\infty} \frac{4^n z^{n/2}}{n!} c_{n/2}^\dagger \quad \text{and} \quad c_{n/2}^\dagger = \int x^{2^{-1}(n+1)-\sigma} dG_\theta(x)$$

and $c_{n/2}^\dagger$ is referred to as a ‘‘half-moment’’.

However,

$$c_{n/2}^\dagger = \frac{(1-\theta)^\sigma}{\Gamma(\sigma)} \int_0^\infty x^{2^{-1}(n+1)-\sigma} e^{\theta x} x^{\sigma-1} e^{-x} dx = \frac{\Gamma(2^{-1}n+2^{-1})}{\Gamma(\sigma)} \frac{(1-\theta)^{\sigma-1/2}}{(1-\theta)^{n/2}},$$

and by Stirling formula,

$$\begin{aligned} \frac{\Gamma(2^{-1}n+2^{-1})}{n!} &\leq C \frac{\sqrt{\pi(n-1)} \left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}}{e^{\frac{n-1}{2}} \sqrt{2\pi n} \left(\frac{n}{e}\right)^n} \leq C e^{\frac{n}{2}} 2^{-\frac{n}{2}} \frac{(n-1)^{n/2} (n-1)^{-1/2}}{n^{n/2}} \\ &\leq C e^{\frac{n}{2}} 2^{-\frac{n}{2}} \frac{(n-1)^{-1/2}}{n^{n/2}} \leq C 2^{-\frac{n}{2}} \frac{n^{-1/4}}{\sqrt{n!}}. \end{aligned}$$

Therefore,

$$B_{\text{III}}(z) \leq C (1-\theta)^{\sigma-1/2} \sum_{n=0}^{\infty} \frac{4^n z^{n/2} 2^{-\frac{n}{2}}}{(1-\theta)^{n/2}} \frac{1}{\sqrt{n!}} = C (1-\theta)^{\sigma-1/2} Q^* \left(\frac{8z}{1-\theta} \right), \quad (79)$$

where $Q^*(z) = \sum_{n=0}^{\infty} \frac{z^{n/2}}{\sqrt{n!}}$. By definition (8.01) and identify (8.07) in Chapter 8 of [Olver \(1974\)](#),

$$Q^*(z) = \sqrt{2} (2\pi z)^{1/4} \exp(2^{-1}z) (1 + O(z^{-1})). \quad (80)$$

Combining (78), (79) and (80) gives

$$\int_{A_{2,z}} \tilde{w}^2(z, x) dG_\theta(x) \leq C (1-\theta)^{\sigma-1/2} z^{\frac{1}{2}-\sigma} \left(\frac{z}{1-\theta} \right)^{1/4} \exp \left(\frac{4z}{1-\theta} \right)$$

for all positive and sufficiently large z . Recall (77). Thus, when $1-\theta > 0$, $\sigma > 0$ and z is

positive and sufficiently large,

$$\begin{aligned}\mathbb{E} [\tilde{w}^2(z, Z)] &\leq \int_{A_{1,z}} \tilde{w}^2(z, x) dG_\theta(x) + \int_{A_{2,z}} \tilde{w}^2(z, x) dG_\theta(x) \\ &\leq C \left(z^{-\sigma} + \left(\frac{z}{1-\theta} \right)^{3/4-\sigma} \exp\left(\frac{4z}{1-\theta} \right) \right) \\ &\leq C \left(\frac{z}{1-\theta} \right)^{3/4-\sigma} \exp\left(\frac{4z}{1-\theta} \right).\end{aligned}$$

Thus, (76) holds. ■

Now we show the theorem. Define

$$w(t, x) = \Gamma(\sigma) \sum_{n=0}^{\infty} \frac{(-tx)^n \cos(2^{-1}\pi n + t\xi(\theta_0))}{n! \Gamma(n + \sigma)} \quad \text{for } t \geq 0 \text{ and } x > 0.$$

Set $S_m(t) = m^{-1} \sum_{i=1}^m (w(t, z_i) - \mathbb{E}[w(t, z_i)])$. Then

$$K(t, x; \theta_0) = \int_{[-1,1]} w(ts, x) \omega(s) ds.$$

Recall $V_m(\hat{\varphi}) = \mathbb{V}[\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})]$. Since $|w(t, x)| \leq \Gamma(\sigma) \tilde{w}(t, x)$ uniformly in (t, x) , (76) implies, for positive and sufficiently large t ,

$$\begin{aligned}V_m(\hat{\varphi}) &\leq m^{-2} \Gamma^2(\sigma) \sum_{i=1}^m \mathbb{E}[\tilde{w}^2(t, x)] \\ &\leq \frac{C}{m^2} \sum_{i=1}^m \left(\frac{t}{1-\theta_i} \right)^{3/4-\sigma} \exp\left(\frac{4t}{1-\theta_i} \right) \leq \frac{1}{m} V_{\text{III}}^{(m)},\end{aligned}$$

where

$$V_{\text{III}}^{(m)} = \frac{C}{m} \exp\left(\frac{4t}{u_{3,m}} \right) \sum_{i=1}^m \left(\frac{t}{1-\theta_i} \right)^{3/4-\sigma}.$$

and $u_{3,m} = \min_{1 \leq i \leq m} \{1 - \theta_i\}$. Recall $\tilde{p}_m(\lambda) = \Pr(|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\theta})| \geq \lambda)$. So, $\Pr(|S_m(t)| \geq \lambda) \leq \lambda^{-2} V_{\text{III}}^{(m)}$ and

$$\tilde{p}_m(\lambda) \leq \Pr\left(|S_m(t)| \geq \frac{\zeta(\theta_0)\lambda}{2\|\omega\|_\infty}\right) \leq \frac{\|\omega\|_\infty^2}{\zeta^2(\theta_0)} \frac{1}{\lambda^2} V_{\text{III}}^{(m)},$$

completing the proof.

E.3 Proof of Theorem 9

Recall

$$V_{\text{III}}^{(m)} = \frac{C}{m} \exp\left(\frac{4t}{u_{3,m}} \right) \sum_{i=1}^m \left(\frac{t}{1-\theta_i} \right)^{3/4-\sigma}$$

and (44), i.e.,

$$\Pr(|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\mu})| \geq \lambda) \leq \frac{1}{\lambda^2} V_{\text{III}}^{(m)},$$

where $u_{3,m} = \min_{1 \leq i \leq m} \{1 - \theta_i\}$ and $\theta_i < 1$, we divide the rest of the proof into two cases: $\sigma > 3/4$ or $\sigma \leq 3/4$. If $\sigma > 3/4$ and $\|\boldsymbol{\theta}\| \leq \rho$, then

$$V_{\text{III}}^{(m)} \leq \frac{C}{m} t^{3/4-\sigma} \exp\left(\frac{4t}{u_{3,m}}\right) \left(\max_{1 \leq i \leq m} (1 - \theta_i)\right)^{\sigma-3/4} \leq \frac{C}{m} t^{3/4-\sigma} \exp\left(\frac{4t}{u_{3,m}}\right).$$

So, we can set $t = 4^{-1} u_{3,m} \gamma \log m$ for any fixed $\gamma \in (0, 1]$ to obtain

$$\Pr(|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\mu})| \geq \lambda) \leq \frac{C}{m^{1-\gamma} \lambda^2} (4^{-1} u_{3,m} \gamma \log m)^{3/4-\sigma}, \quad (81)$$

which implies

$$\Pr\left(\frac{|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\mu})|}{\pi_{1,m}} \geq \varepsilon\right) \leq C \varepsilon^{-2} (4^{-1} u_{3,m} \gamma \log m)^{3/4-\sigma} \rightarrow 0 \text{ as } m \rightarrow \infty$$

for any fixed $\varepsilon > 0$ whenever $\pi_{1,m} \geq C m^{(\gamma-1)/2}$ and $u_{3,m} \gamma \log m \rightarrow \infty$.

In contrast, if $\sigma \leq 3/4$, then

$$V_{\text{III}}^{(m)} \leq \frac{C}{m} \left(\frac{t}{u_{3,m}}\right)^{3/4-\sigma} \exp\left(\frac{4t}{u_{3,m}}\right).$$

So, we can still set $t = 4^{-1} u_{3,m} \gamma \log m$ for any fixed $\gamma \in (0, 1)$, which implies

$$\Pr\left(\frac{|\hat{\varphi}_m(t, \mathbf{z}) - \varphi_m(t, \boldsymbol{\mu})|}{\pi_{1,m}} \geq \varepsilon\right) \leq \frac{C \varepsilon^{-2} (4^{-1} \gamma \log m)^{3/4-\sigma}}{m^{\gamma'-\gamma}} \rightarrow 0 \text{ as } m \rightarrow \infty$$

whenever $\pi_{1,m} \geq C m^{(\gamma'-1)/2}$ for any $\gamma' > \gamma$. Recall

$$\psi(t, \mu; \mu_0) = \int_{[-1,1]} \cos(ts(\xi(\theta_0) - \xi(\theta))) \omega(s) ds,$$

which converges to 0 as $m \rightarrow \infty$ when $\lim_{m \rightarrow \infty} t \min_{i \in I_{1,m}} |\xi(\theta_0) - \xi(\theta_i)| = \infty$. Noticing $u_{3,m} = \min_{1 \leq i \leq m} \xi^{-1}(\theta_i)$, we have shown the claim.

F Additional simulation results

This section presents the performances of the “MR” estimator of [Meinshausen and Rice \(2006\)](#) and the proposed estimator “New” when they are applied to each of the five families, i.e., Cauchy, Laplace, Poisson, Negative Binomial and Gamma families.

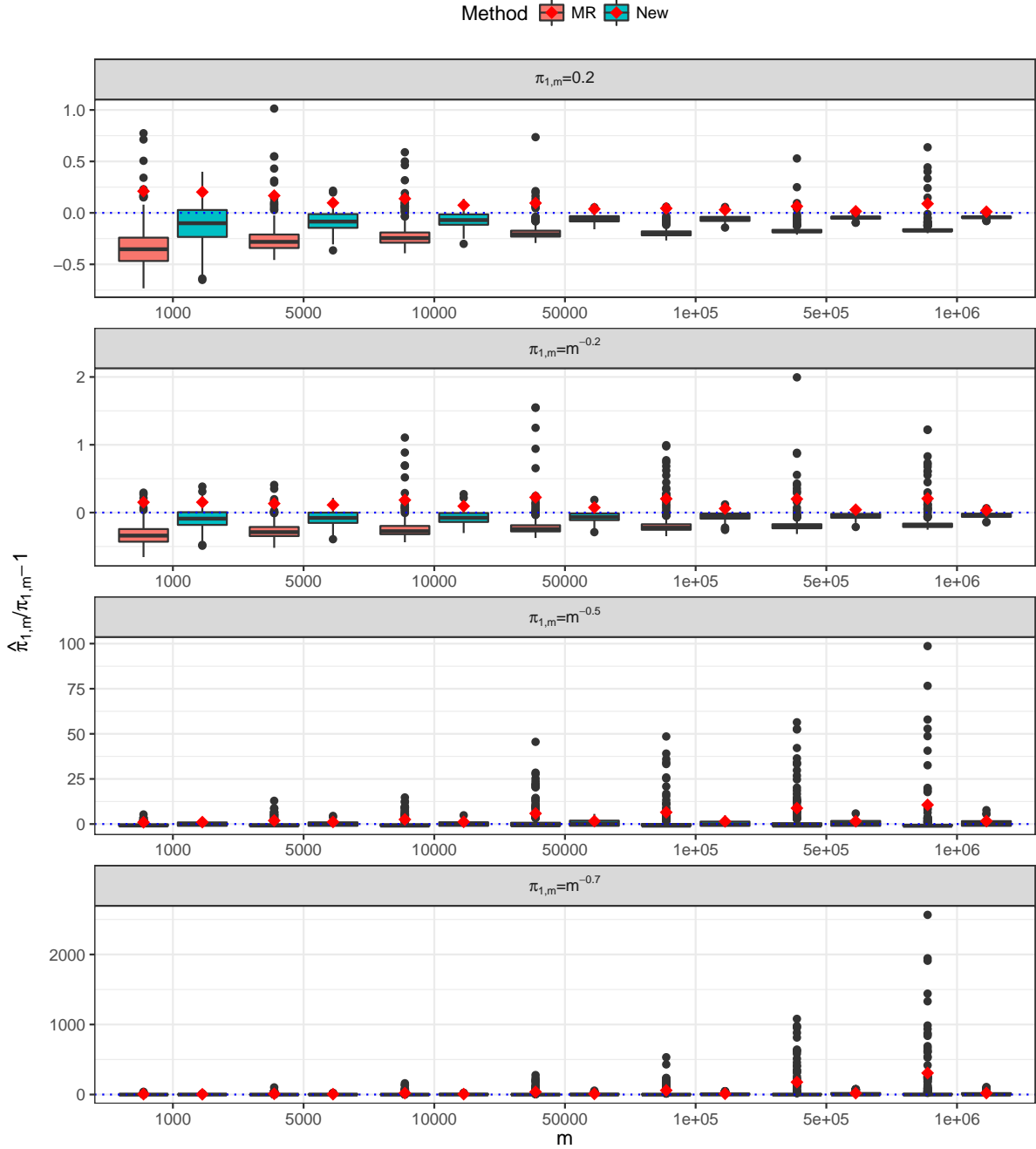


Figure F.1: Boxplot of the excess $\tilde{\delta}_m = \hat{\pi}_{1,m}\pi_{1,m}^{-1} - 1$ of the “MR” estimator of [Meinshausen and Rice \(2006\)](#) and of the proposed estimator “New” when they are applied to Cauchy family. The boxplot for the proposed estimator is the right one in each pair of boxplots for each m . The thick horizontal line and the diamond in each boxplot are respectively the mean and standard deviation of $\tilde{\delta}_m$, and the dotted horizontal line in each panel corresponding to a setting of $\pi_{1,m}$ is the reference for $\tilde{\delta}_m = 0$.

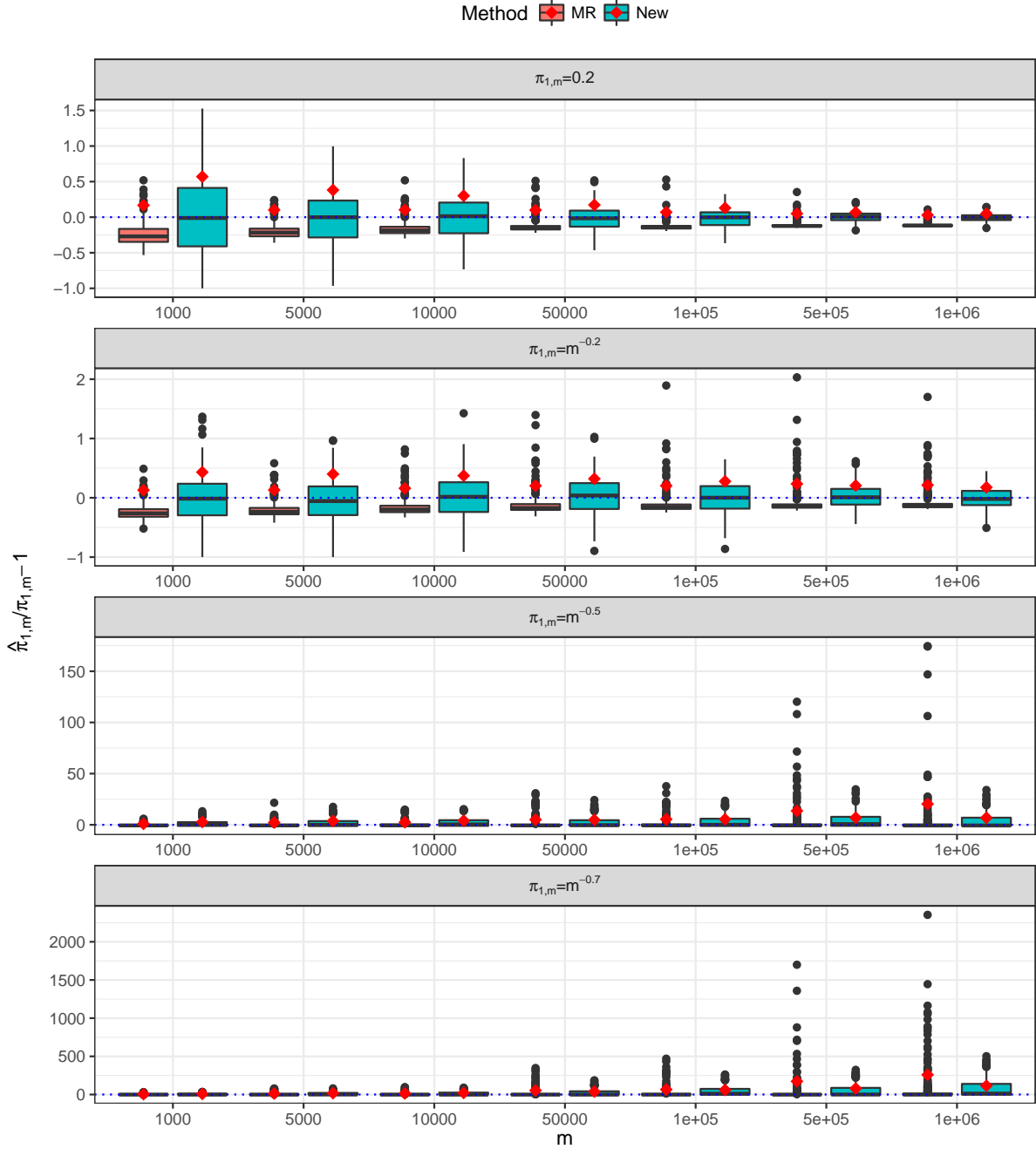


Figure F.2: Boxplot of the bias $\tilde{\delta}_m = \hat{\pi}_{1,m}\pi_{1,m}^{-1} - 1$ of the “MR” estimator of [Meinshausen and Rice \(2006\)](#) and of the proposed estimator “New” when they are applied to Laplace family. The boxplot for the proposed estimator is the right one in each pair of boxplots for each m . The thick horizontal line and the diamond in each boxplot are respectively the mean and standard deviation of $\tilde{\delta}_m$, and the dotted horizontal line in each panel corresponding to a setting of π_m is the reference for $\tilde{\delta}_m = 0$.

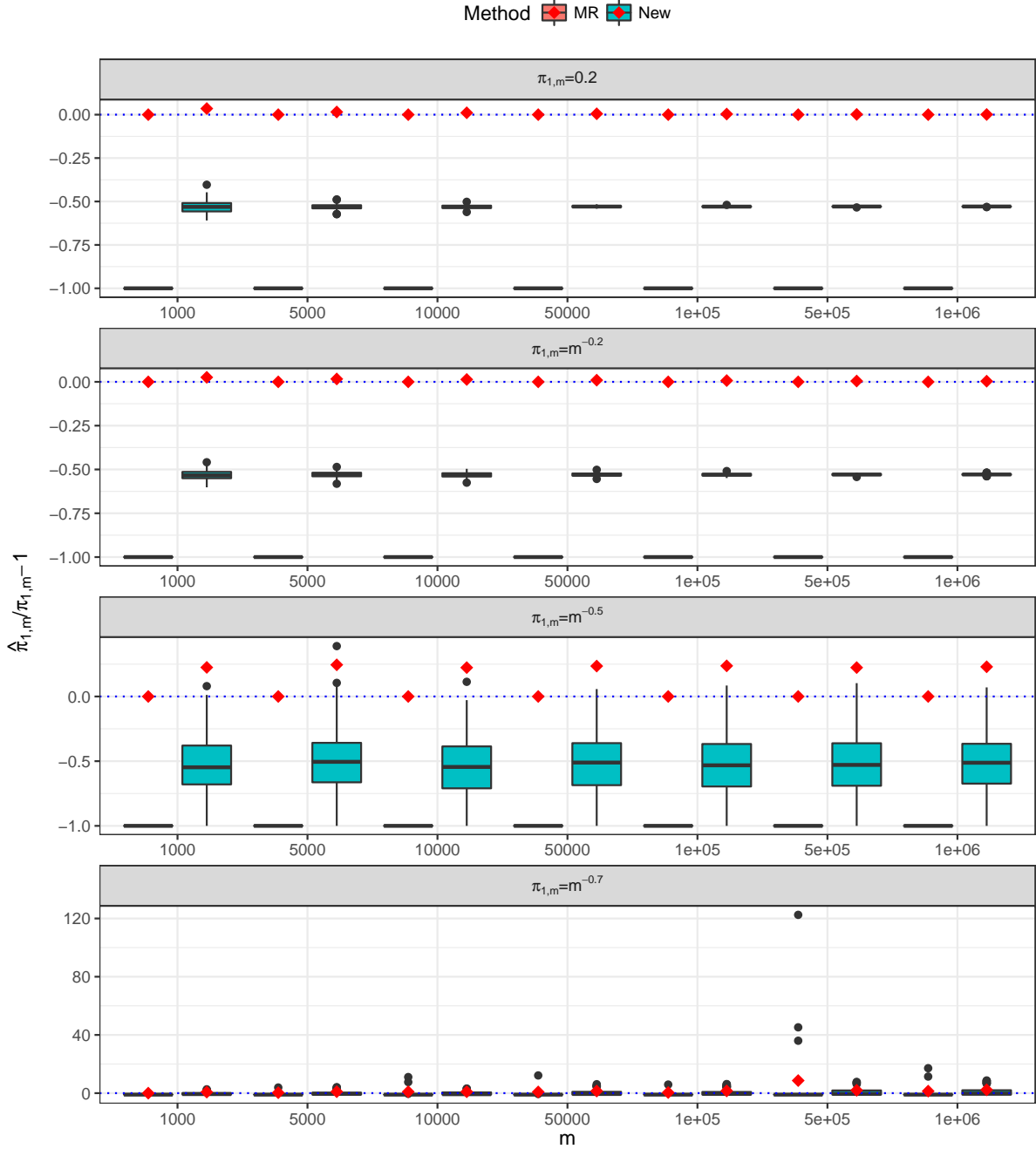


Figure F.3: Boxplot of the bias $\tilde{\delta}_m = \hat{\pi}_{1,m}\pi_{1,m}^{-1} - 1$ of the “MR” estimator of [Meinshausen and Rice \(2006\)](#) and of the proposed estimator “New” when they are applied to Negative Binomial family. The boxplot for the proposed estimator is the right one in each pair of boxplots for each m . The thick horizontal line and the diamond in each boxplot are respectively the mean and standard deviation of $\tilde{\delta}_m$, and the dotted horizontal line in each panel corresponding to a setting of $\pi_{1,m}$ is the reference for $\tilde{\delta}_m = 0$.

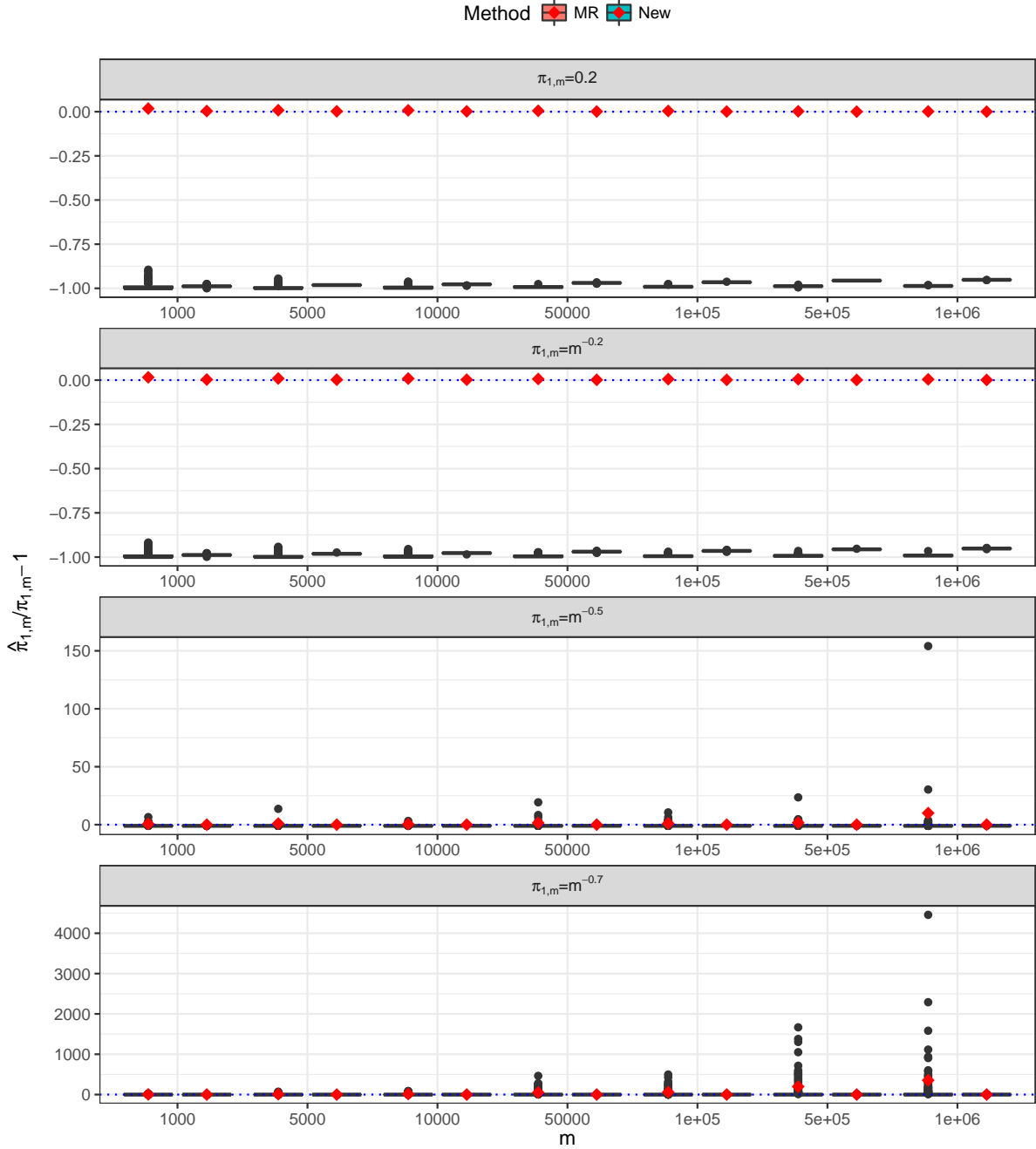


Figure F.4: Boxplot of the bias $\tilde{\delta}_m = \hat{\pi}_{1,m}\pi_{1,m}^{-1} - 1$ of the “MR” estimator of [Meinshausen and Rice \(2006\)](#) and of the proposed estimator “New” when they are applied to Chi-square family. The boxplot for the proposed estimator is the right one in each pair of boxplots for each m . The thick horizontal line and the diamond in each boxplot are respectively the mean and standard deviation of $\tilde{\delta}_m$, and the dotted horizontal line in each panel corresponding to a setting of $\pi_{1,m}$ is the reference for $\tilde{\delta}_m = 0$.

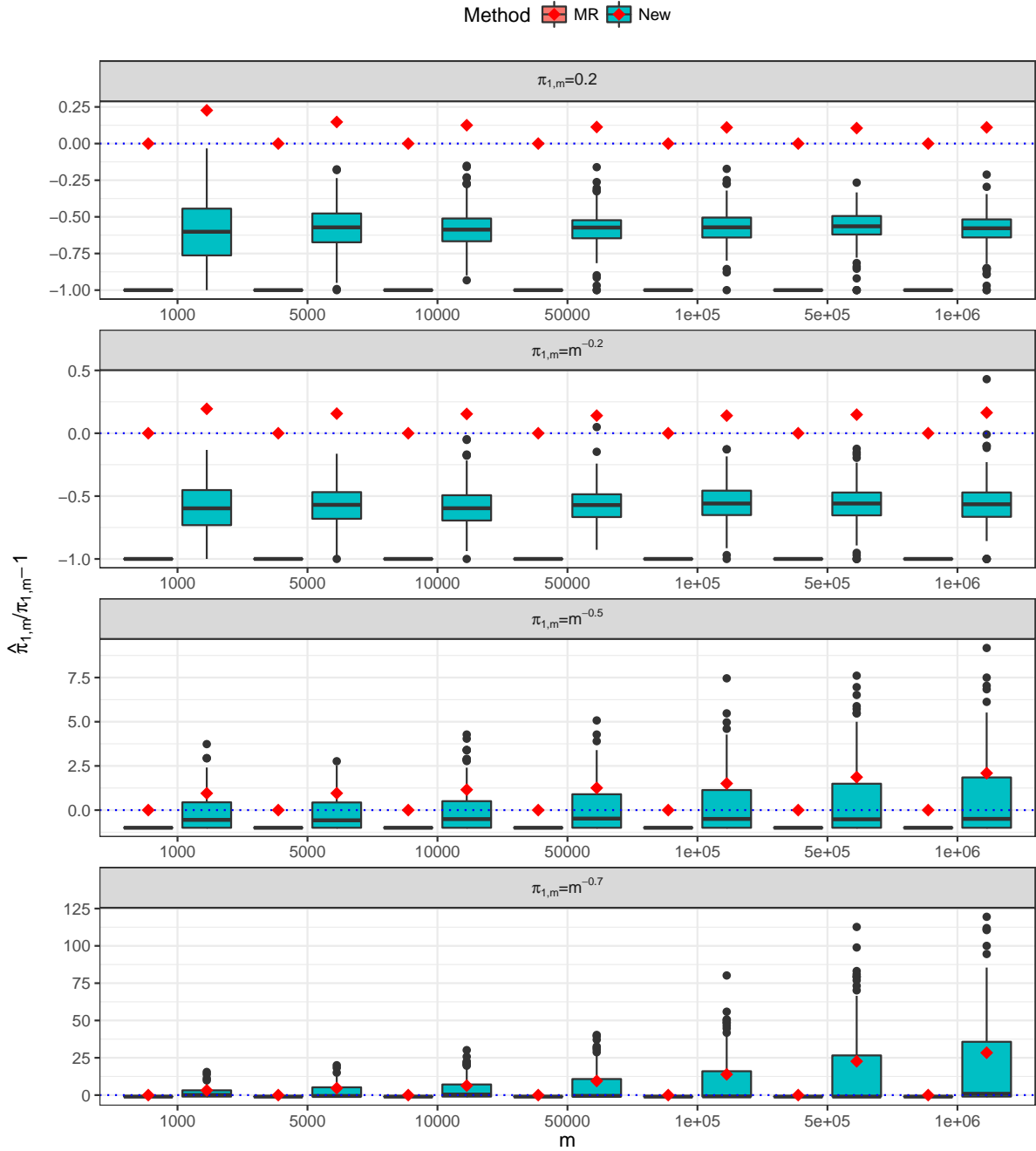


Figure F.5: Boxplot of the bias $\tilde{\delta}_m = \hat{\pi}_{1,m}\pi_{1,m}^{-1} - 1$ of the “MR” estimator of [Meinshausen and Rice \(2006\)](#) and of the proposed estimator “New” when they are applied to Poisson family. The boxplot for the proposed estimator is the right one in each pair of boxplots for each m . The thick horizontal line and the diamond in each boxplot are respectively the mean and standard deviation of $\tilde{\delta}_m$, and the dotted horizontal line in each panel corresponding to a setting of $\pi_{1,m}$ is the reference for $\tilde{\delta}_m = 0$.